Math4230 Exercise 1

1. Let $C \subset \mathbb{R}^n$. C is called a *cone* if $\lambda x \in C$ whenever $\lambda > 0$ and $x \in C$. Show that a cone C is convex if and only if $C + C \subseteq C$, where $C + C := \{x + y | \ x, y \in C\}$.

2. Show that the interior and closure of a convex set is also convex.

3. Show that the image and inverse image of a convex set under a linear transformation is also a convex set.

4. (a) A *perspective function* is a function $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ such that

$$f(x,t) = \begin{bmatrix} x_1/t \\ x_2/t \\ \vdots \\ x_n/t \end{bmatrix}$$

   where $x \in \mathbb{R}^n$ and $t > 0$.
   Show that the $f(C)$ is convex if C is convex and f is a perspective function.

   (b) Show that $f^{-1}(C)$ is convex if C is convex and f is a perspective function.

   (c) A *linear fractional function* is a function of the form

$$h(x) = \frac{Ax + b}{c^T x + d}$$

   where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. The domain of f is assumed to be $\{x \mid c^T x + d > 0\}$.
   Show that $h(C)$ is convex if C is convex and h is a linear fractional function.