1.5 Projection to Convex Sets

Given a set $C \subseteq \mathbb{R}^n$, the distance of a point to C is defined by

$$d(x;C) := \inf\{||x-y|| \mid y \in C\}$$

For closed convex sets, an important consequence is the following projection property.

Proposition:(Projection Property) Let C be a nonempty, closed convex subset of $\mathbb{R}^n$. For each $x \in \mathbb{R}^n$, there exists an unique $w \in C$ such that

$$||x-w|| = d(x;C)$$

w is called the projection of x to C, and is denoted by $P_C(x)$.

Proof. By definition of $d(x;C)$, there exists $w_k \in C$ such that

$$d(x;C) \leq ||x-w_k|| < d(x;C) + \frac{1}{k}$$

It follows that $\{w_k\}$ is a bounded sequence. Hence it has a converging subsequence $\{w_{k_l}\}$ which converges to a point w. Since C is closed, $w \in C$.
Considering the limit of

$$d(x;C) \leq ||x-w_{k_l}|| < d(x;C) + \frac{1}{k_l}$$

Hence $d(x;C) = ||x-w||$.
Now suppose $w_1 \neq w_2 \in C$ satisfy

$$||x-w_1|| = ||x-w_2|| = d(x;C)$$

Then we have,

$$2||x-w_1||^2 = ||x-w_1||^2 + ||x-w_2||^2 = 2||x - \frac{w_1+w_2}{2}||^2 + \frac{||w_1-w_2||^2}{2}$$

Since C is convex, $\frac{w_1+w_2}{2} \in C$. This gives,

$$||x - \frac{w_1+w_2}{2}||^2 = ||x-w_1||^2 - \frac{||w_1-w_2||^2}{4} < ||x-w_1||^2 = d(x;C)^2$$

But since C is convex, $\frac{w_1+w_2}{2} \in C$, this is a contradiction. □

Proposition: Let C be a nonempty, closed convex set, then $w = P_C(x)$ if and only if

$$\langle x-w, u-w\rangle \leq 0, \ \forall u \in C$$

Proof. Suppose $w = P_C(x)$.
Let $u \in C,\ \lambda \in (0,1)$. Since C is convex, $\lambda u + (1-\lambda)w \in C$. Then

$$||x-w||^2 = d(x;C)^2 \leq ||x-w-\lambda(u-w)||^2 = ||x-w||^2 - 2\lambda\langle x-w, u-w\rangle + \lambda^2||u-w||^2.$$

That is

$$2\langle x-w, u-w\rangle \leq \lambda||u-w||^2$$

Letting $\lambda \to 0^+$, we have

$$\langle x-w, u-w\rangle \leq 0$$

Conversely, suppose

$$\langle x-w, u-w\rangle \leq 0,\ \forall u \in C$$

Then

$$\begin{aligned} ||x-u||^2 &= ||x-w||^2 + 2\langle x-w, w-u\rangle + ||w-u||^2 \\ &\geq ||x-w||^2 - 2\langle x-w, u-w\rangle \geq ||x-w||^2 \end{aligned}$$

Hence $||x-w|| \leq ||x-u||$ for all $u \in C$ and $w = P_C(x)$. □

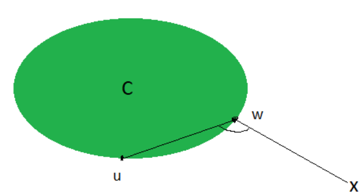

Figure 1: Projection to a convex set

2 Subdifferential Calculus

2.1 Convex Separation

The separating theorems are of fundamental importance in convex analysis and optimization. This section provides some of the useful results.

Definition:(Hyperplane Separation) Two sets C_1, C_2 are said to be separated by a hyperplane if there exists $a \neq 0$ such that

$$\sup_{x \in C_1} \langle a, x \rangle \leq \inf_{y \in C_2} \langle a, y \rangle$$

C_1, C_2 are said to be strictly separated if there exists $a \neq 0$ such that

$$\sup_{x \in C_1} \langle a, x \rangle < \inf_{y \in C_2} \langle a, y \rangle$$

If x is a relative boundary point of C, a hyperplane that separates C and $\{x\}$ is called a supporting hyperplane at x.

We will focus on the separation of two convex sets. To proof the existence of such separation, we start with two lemmas.
Lemma: Let C be an nonempty, closed convex set and $\overline{x} \notin C$. Then there exists nonzero a such that

$$\sup_{x \in C} \langle a, x \rangle < \langle a, \overline{x} \rangle$$

Proof. Let $w = P_C(\overline{x})$ (which exists by the projection property). Then

$$\langle \overline{x} - w, x \rangle \leq \langle \overline{x} - w, w \rangle \text{ for all } x \in C.$$

Let $a = \overline{x} - w \neq 0$, then

$$\langle a, x \rangle \leq \langle a, w \rangle = \langle a, \overline{x} \rangle - ||\overline{x} - w||^2 < \langle a, \overline{x} \rangle$$

□

Lemma: Let C be a nonempty, convex subset of $\mathbb{R}^n$ with $x \in \overline{C} \backslash \mathrm{ri}(C)$. Then there exists $\{x_k\}$ such that $x_k \to x$ while $x_k \notin \overline{C}$ for all k.

Proof. Since $\mathrm{ri}(C)$ is nonempty, pick $x_0 \in \mathrm{ri}(C)$.
Let $x_k = \frac{k+1}{k} x - \frac{x_0}{k}$.
Clearly, $x_k \to x$. It remains to show that $x_k \notin \overline{C}$. Suppose otherwise, then by the Line Segment property,

$$x = \frac{1}{k+1} x_0 + \frac{k}{k+1} \left(\frac{k+1}{k} x - \frac{x_0}{k} \right) \in \mathrm{ri}(C)$$

This is a contradiction. Hence $x_k \notin \overline{C}$ for all k.

□

Theorem:(Supporting Hyperplane Theorem) Let C be a nonempty, convex set. Suppose $\overline{x} \in$rel $\partial C = \overline{C} \backslash \text{ri}(C)$. Then there exists $a \neq 0$ such that

$$\sup_{x \in \overline{C}} \langle a, x \rangle \leq \langle a, \overline{x} \rangle$$

Proof. Since $\overline{x} \in$rel ∂C. Then there exists $x_k \notin \overline{C}$ with $x_k \to \overline{x}$.
By lemma, there exists $a_k \neq 0$ such that

$$\sup_{x \in \overline{C}} \langle a_k, x \rangle < \langle a_k, x_k \rangle$$

By dividing $\|a_k\|$, we may assume $\{a_k\}$ is bounded.
Since $\{a_k\}$ is bounded, it has a converging subsequence.
Without loss of generality (considering the subsequence), we may assume that $a_k \to a \neq 0$
Taking the limit, we have for all $x \in \overline{C}$

$$\langle a, x \rangle \leq \langle a, \overline{x} \rangle$$

□

Theorem:(Separating Hyperplane Theorem) Let C_1, C_2 be two convex sets. Suppose $C_1 \cap C_2 = \emptyset$. Then there exists a hyperplane that separates C_1 and C_2.

Proof. Consider $C := C_1 - C_2$. Since $C_1 \cap C_2 = \emptyset$, $0 \notin C$.
There are two cases:
Case (1): $0 \in \overline{C}$.
By the supporting hyperplane theorem, there exists $a \neq 0$ such that

$$\langle a, x \rangle \leq \langle a, 0 \rangle = 0, \text{ for all } x \in C$$

That is

$$\langle a, x_1 \rangle \leq \langle a, x_2 \rangle$$

Case (2): $0 \notin \overline{C}$
The result follows directly from the previous lemma. □

In order to get strict separation, we need more assumptions.

Theorem:(Strict Hyperplane Separation) Let C_1, C_2 be nonempty, closed convex sets with $C_1 \cap C_2 = \emptyset$. Suppose at least one of the two sets is also bounded. Then there exists $a \neq 0$ such that

$$\sup_{x_1 \in C_1} \langle a, x_1 \rangle < \inf_{x_2 \in C_2} \langle a, x_2 \rangle$$

Proof. Let $C := C_1 - C_2$. Then C is a nonempty, closed convex set with $0 \notin C$. Then there exists $a \neq 0$ such that

$$\gamma := \sup_{x \in C} \langle a, x \rangle < 0$$

Then for all $x_1 \in C_1$, $x_2 \in C_2$, we have $\langle a, x_1 \rangle \leq \gamma + \langle a, x_2 \rangle$. Then

$$\sup_{x_1 \in C_1} \langle a, x_1 \rangle \leq \gamma + \inf_{x_2 \in C_2} \langle a, x_2 \rangle < \inf_{x_2 \in C_2} \langle a, x_2 \rangle$$

□