THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MMAT 5340 Probability and Stochastic Analysis

Suggested Solution of Homework 1

1.18 $\binom{7}{3}\binom{8}{3} = 1960$

1.19 $\binom{7}{3}\binom{8}{3}(6) = 11760$

1.20 $\binom{7}{3}\binom{8}{2}\binom{6}{1} = 5880$

2.45 Note that

$$P(A) = \frac{3\times 3 + 3\times 3}{6^2} = \frac{1}{2}, \quad P(B) = \frac{3}{6} = \frac{1}{2} \quad \text{and} \quad P(A\cap B) = \frac{3\times 3}{6^2} = \frac{1}{4}.$$

Hence

$$P(A \mid B) = \frac{P(A\cap B)}{P(B)} = \frac{1/4}{1/2} = \frac{1}{2},$$

and

$$P(B \mid A) = \frac{P(A\cap B)}{P(A)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

3.23 The distribution of X is given by

$$P(X = k) = \frac{(1)(10-k)}{\binom{10}{2}} = \frac{10-k}{45}, \qquad 1 \le k \le 10.$$

3.24

$$\mathbf{E}X = \sum_{k=1}^{10} kP(X=k) = \sum_{k=1}^{10} \frac{k(10-k)}{45} = \frac{11}{3}.$$

$$\mathbf{E}X^2 = \sum_{k=1}^{10} k^2P(X=k) = \sum_{k=1}^{10} \frac{k^2(10-k)}{45} = \frac{55}{3}.$$

Hence

$$\mathrm{Var}X = \mathbf{E}X^2 - (\mathbf{E}X)^2 = \frac{55}{3} - \left(\frac{11}{3}\right)^2 = \frac{44}{9}.$$

3.25

$$P(X \le 3) = P(X=1) + P(X=2) + P(X=3) = \frac{8}{15}.$$

4.21 Since each try is independent to another,

$$\begin{aligned} P(\text{Exactly 10 tries}) &= P(\text{1 success in first 9 tries}) \cdot P(\text{success in 10th try}) \\ &= \binom{9}{1}\left(\frac{2}{6}\right)^1\left(\frac{4}{6}\right)^8 \cdot \left(\frac{2}{6}\right)^1 \\ &= \frac{256}{6561}. \end{aligned}$$

4.22 Let X and Y be the number of accidents this year from good and bad drivers, respectively. Then $X \sim \text{Bin}(5000, 0.0001)$ and $Y \sim \text{Bin}(3000, 0.0003)$. Note that $\lambda_1 = N_1 p_1 = 0.5$ and $\lambda_2 = N_2 p_2 = 0.9$. By Poisson approximation, X and Y are approximately distributed as $\text{Poi}(\lambda_1)$ and $\text{Poi}(\lambda_2)$, respectively. Since X and Y are independent, $X + Y$ is approximately distributed as $\text{Poi}(\lambda)$, where $\lambda = \lambda_1 + \lambda_2 = 0.5 + 0.9 = 1.4$. Hence

$$\begin{aligned} P(\text{at least two accidents}) &= P(X+Y \geq 2) \\ &= 1 - P(X+Y=0) - P(X+Y=1) \\ &\approx 1 - \frac{1.4^0 e^{-1.4}}{0!} - \frac{1.4^1 e^{-1.4}}{1!} \\ &= 1 - 2.4e^{-1.4} \\ &\approx 0.4082. \end{aligned}$$

4.23 Let X_1 be the number of trials needed to get the first successful trial, and for $2 \leq i \leq 4$, let X_i be the number of trials needed after the $(i-1)$-th successful trial to get the next successful trial. Then X_1, X_2, X_3, X_4 are independent with

$$X_i \sim \text{Geo}(0.4) \quad ,1 \leq i \leq 4 \quad \text{and} \quad X_1 + X_2 + X_3 + X_4 = X.$$

Hence

$$\mathbf{E}X = \mathbf{E}X_1 + \mathbf{E}X_2 + \mathbf{E}X_3 + \mathbf{E}X_4 = \frac{1}{0.4} \times 4 = 10,$$

and

$$\text{Var}X = \text{Var}X_1 + \text{Var}X_2 + \text{Var}X_3 + \text{Var}X_4 = \frac{1-0.4}{0.4^2} \times 4 = 15.$$

5.18

$$\begin{aligned} P(X \leq 3 \mid X \geq 2) &= \frac{P(2 \leq X \leq 3)}{P(X \geq 2)} \\ &= \frac{\int_2^3 f(x)dx}{\int_2^\infty f(x)dx} \\ &= \frac{\int_2^3 2x^{-3}dx}{\int_2^\infty 2x^{-3}dx} \\ &= \frac{-x^{-2}\big|_2^3}{-x^{-2}\big|_2^\infty} \\ &= \frac{5}{9}. \end{aligned}$$

5.45 Since $X \sim \text{Exp}(1/2)$, X only takes non-negative values. For $0 \leq a \leq b$,

$$P(a \leq 3X^4 \leq b) = P((a/3)^{1/4} \leq X \leq (b/3)^{1/4}) = \int_{(a/3)^{1/4}}^{(b/3)^{1/4}} \frac{1}{2} e^{-\frac{1}{2}x} dx.$$

Now by the change of variable $x = (y/3)^{1/4}$, we have

$$a \leq y \leq b \quad \text{and} \quad dx = (1/4)(y/3)^{-3/4}(1/3) = \frac{1}{12}\left(\frac{y}{3}\right)^{-3/4}.$$

Then the integral takes the form

$$P(a \le 3X^4 \le b) = \int_a^b \frac{1}{2} e^{-\frac{1}{2}(y/3)^{1/4}} \frac{1}{12}\left(\frac{y}{3}\right)^{-3/4} dy = \int_a^b p(y)dy,$$

where

$$p(y) := \begin{cases} \frac{1}{24} e^{-\frac{1}{2}(y/3)^{1/4}} \left(\frac{y}{3}\right)^{-3/4} & \text{if } y > 0, \\ 0 & \text{if } y \le 0, \end{cases}$$

is the density of $3X^4$.

5.46 Since $X \sim \Gamma(3,2)$ and $Y \sim \Gamma(5,2)$, we have X, Y positive and

$$p_X(x) = \frac{2^3}{2!} x^2 e^{-2x}, \quad \text{and} \quad p_Y(y) = \frac{2^5}{4!} y^4 e^{-2y} \quad \text{for } x, y > 0.$$

Then the density of $X+Y$ is given by

$$\begin{aligned} p_{X+Y}(x) &= \int_0^x p_X(y) p_Y(x-y) dy \\ &= \int_0^x \frac{2^3}{2!} y^2 e^{-2y} \frac{2^5}{4!} (x-y)^4 e^{-2(x-y)} dy \\ &= \frac{2^8 e^{-2x}}{2!4!} \int_0^x y^2 (x-y)^4 dy \\ &= \frac{2^8 e^{-2x}}{2!4!} \left(-\frac{1}{5} y^2 (y-x)^5 \Big|_0^x + \frac{2}{5} \int_0^x y(x-y)^5 dy \right) \\ &= \frac{2^8 e^{-2x}}{2!4!} \left(\frac{2}{5} \int_0^x y(x-y)^5 dy \right) \\ &= \frac{2^8 e^{-2x}}{2!4!} \left(\frac{2}{5} \cdot \frac{1}{6} \int_0^x (x-y)^6 dy \right) \\ &= \frac{2^8 e^{-2x}}{2!4!} \left(\frac{2}{5} \cdot \frac{1}{6} \cdot \frac{-1}{7} (x-y)^7 \Big|_0^x \right) \\ &= \frac{2^8 e^{-2x}}{2!4!} \left(\frac{2}{5} \cdot \frac{1}{6} \cdot \frac{1}{7} x^7 \right) \\ &= \frac{2^8}{7!} x^7 e^{-2x}, \end{aligned}$$

for $x > 0$ and equals to zero elsewhere. Hence $X + Y \sim \Gamma(8,2)$.

6.8 Let X be the number of votes Richard gets. Then $X \sim \text{Bin}(100, 0.49)$, so that $\mathbf{E}X = (100)(0.49) = 49$ and $\text{Var}X = (100)(0.49)(0.51) = 24.99$. Now

$$P(\text{Richard wins}) = P(X > 50) = P\left(\frac{X-49}{\sqrt{24.99}} > \frac{1}{\sqrt{24.99}} \right).$$

By Normal approximation, the probability is approximately $P\left(Z > \frac{1}{\sqrt{24.99}}\right)$, where $Z \sim \mathcal{N}(0,1)$. Hence

$$P(\text{Richard wins}) \approx P\left(Z > \frac{1}{\sqrt{24.99}}\right) \approx 0.4207.$$

8.11 Recall that the generating function of a random variable X is given by

$$\varphi_X(s) := \mathbf{E}s^X = \sum_{k=0}^{\infty} P(X=k)s^k.$$

Since X and Y are independent, we have

$$F_{X+2Y+1}(t) = \mathbf{E}s^{(X+2Y+1)} = s\ \mathbf{E}\left(s^X s^{2Y}\right) = s\ \mathbf{E}s^X\mathbf{E}(s^2)^Y = s\varphi_X(t)\varphi_Y(s^2).$$

Since $X \sim \text{Bin}(3, 0.4)$ and $Y \sim \text{Geo}(0.3)$, we have

$$\varphi_X(s) = (0.4s+0.6)^3,$$

and

$$\varphi_Y(s) = \frac{0.3s}{1-0.7s}.$$

Hence,

$$\varphi_{X+2Y+1}(s) = s(0.4s+0.6)^3\frac{0.3s^2}{1-0.7s^2} = (0.4s+0.6)^3\frac{3s^3}{10-7s^2}.$$

8.21 Note that, for $X=-1$, $F_X(t) = e^{-t}$, and for $Y \sim \Gamma(2,2)$, $F_Y(t) = \left(\frac{2}{2-t}\right)^2$. Since X is a constant, X and Y must be independent. Hence

$$F_{X+Y}(t) = F_X(t)F_Y(t) = e^{-t}\left(\frac{2}{2-t}\right)^2 = \frac{4e^{-t}}{(2-t)^2}.$$

Thus the distribution corresponds to $X+Y$.

9.19 (The number 100 is changed to 50.) Let $S = X_1 + \cdots + X_{50} + Y_1 + \cdots + Y_{40}$. Then $\mu = \mathbf{E}S = 50p_X + 40p_Y = 64$. Note that $50 = \mu(1-\delta)$, where $\delta = \frac{7}{32}$. Hence, by Chernov's inequality,

$$\begin{aligned} P(S \le 50) &= P(S \le \mu(1-\delta)) \\ &\le \exp\left(-\frac{\delta^2\mu}{2}\right) \\ &= \exp(-\frac{49}{32}) \\ &\approx 0.2163. \end{aligned}$$

10.13 Note that the triangle can also be described as

$$\{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1, 0 \le y \le x\} = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1, y \le x \le 1\}.$$

So, the marginal density of Y is given by

$$p_Y(y) = \int p(x,y)dx = \begin{cases} \displaystyle\int_y^1 2dx = 2(1-y) & \text{if } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$p_{X\,|\,Y}(x\,|\,y) = \frac{p(x,y)}{p_Y(y)} = \begin{cases} \dfrac{1}{1-y} & \text{if } 0 \le y < 1,\ \ y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for $0 \le y < 1$,

$$\begin{aligned} \mathbf{E}(X^2\,|\,Y=y) &= \int x^2 p_{X\,|\,Y}(x\,|\,y)dx \\ &= \int_y^1 \frac{x^2}{1-y}dx \\ &= \left.\frac{x^3}{3(1-y)}\right|_y^1 \\ &= \frac{1}{3}(y^2+y+1). \end{aligned}$$