Total Variation in Image Analysis (The Homo Erectus Stage?)

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Outline

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- In mathematics: the Plateau problem of minimal surfaces, i.e. surfaces of minimal area with a given boundary

- In image analysis: denoising, image reconstruction, segmentation...
- An ubiquitous prior for many image processing tasks.

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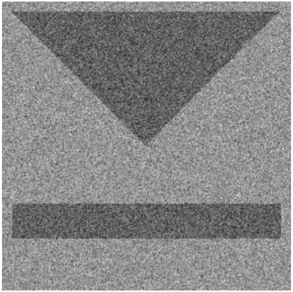
Denoising

Determine an unknown image from a noisy observation.



Methods

All methods based on some statistical inference.

- Fourier/Wavelets
- Markov Random Fields
- Variational and Partial Differential Equations methods
- ...

We focus on variational and PDE methods.

A simple corruption model

- A digital image u of size $N \times M$ pixels, corrupted by Gaussian white noise of variance σ^2
- write it as observed image $u_0 = u + \eta$, $\|u - u_0\|^2 = \sum_{ij}(u_{ij} - u_{0ij})^2 = NM\sigma^2$ (noise variance = σ^2), $\sum_{ij} u_{ij} = \sum_{ij} u_{0ij}$ (zero mean noise).
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Recovery

- The problem: Find u such that

$$\|u - u_0\|^2 = NM\sigma^2, \quad \sum_{ij} u_{ij} = \sum_{ij} u_{0ij} \tag{1}$$

  is not well-posed. Many solutions possible.

- In order to recover u, extra information is needed, e.g. in the form of a prior on u.
- For images, smoothness priors often used.
- Let Ru a digital gradient of u, Then find smoothest u that satisfy constraints (1), the smoothest meaning with smallest

$$T(u) = \|Ru\| = \sqrt{\sum_{ij} |Ru|_{ij}^2}.$$

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Outline

Tikhonov regularization

- It can be show that this is equivalent to minimize

$$E(u) = ||Ku - u_0||^2 + \lambda ||Ru||^2$$

for a $\lambda = \lambda(\sigma)$ (Wahba?).

- $E(u)$ minimizaton can be derived from a Maximum a Posteriori formulation

$$\underset{u}{\text{Arg.max}} p(u|u_0) = \frac{p(u_0|u)p(u)}{p(u_0)}$$

- Rewriting in a continuous setting:

$$E(u) = \int_\Omega (Ku - u_o)^2 \, dx + \lambda \int_\Omega |\nabla u|^2 \, dx$$

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How to solve?

- Solution satisfies the Euler-Lagrange equation for E:

$$K^*\left(Ku - u_0\right) - \lambda \Delta u = 0.$$

  (K^* is the adjoint of K)
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Tikhonov example

Denoising example, $K = Id$.

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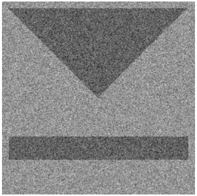
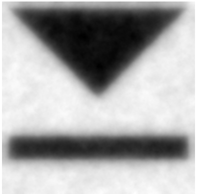


$\lambda = 50$



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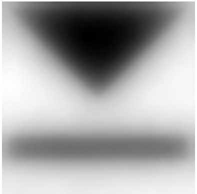

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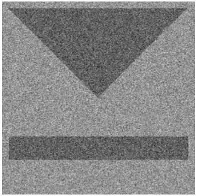

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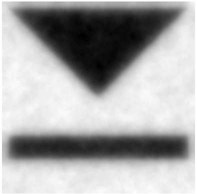

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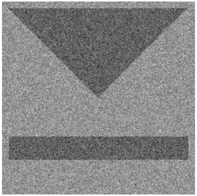

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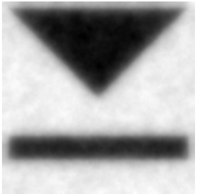

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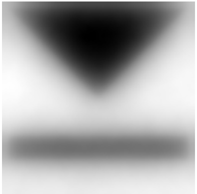

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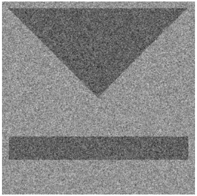

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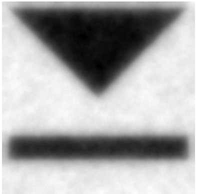

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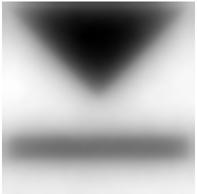

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Outline

- Set $\Omega = [-1, 1]$, a a real number and u the step-edge function

$$u(x) = \begin{cases} 0 & x \leq 0 \\ a & x > 0 \end{cases}$$

- Not differentiable at 0, but forget about it and try to compute

$$\int_{-1}^{1} |u'(x)|^2 \, dx.$$

- Around 0 "approximate" $u'(x)$ by

$$\frac{u(h) - u(-h)}{2h}, \quad h > 0, \text{small}$$

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- with this finite difference approximation

$$u'(x) \approx \frac{a}{2h}, \quad x \in [-h, h]$$

- then

$$\begin{aligned}\int_{-1}^{1} |u'(x)|^2\,dx &= \int_{-1}^{-h} |u'(x)|^2\,dx + \int_{-h}^{h} |u'(x)|^2\,dx + \int_{h}^{1} |u'(x)|^2\,dx \\ &= 0 + 2h \times \left(\frac{a}{2h}\right)^2 + 0 \\ &= \frac{a^2}{2h} \to \infty, \quad h \to 0\end{aligned}$$

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- What went “wrong”: the square:

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- When $p \le 1$ this is finite! Edges can survive here!
- Quite ugly when $p < 1$ (but not uninteresting)
- When $p = 1$, this is the Total Variation of u.

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$$J(u) = \int_\Omega |\nabla u| \, dx, \quad |\nabla u| = \sqrt{\sum_{i=1}^n u_{x_i}^2}.$$

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- Expected: when minimizing $J(u)$ with other constraints, edges are less penalized that with Tikhonov.
- Indeed edges are “naturally present” in bounded variation functions. In fact: functions of bounded variations can be decomposed in
  - smooth parts, ∇u well defined,
  - Jump discontinuities (our edges)
  - something else (Cantor part) which can be nasty...
- The functions that do not possess this nasty part form a subspace of $BV(\Omega)$ called $SBV(\Omega)$, The Special functions of Bounded Variation, (used for instance when studying Mumford-Shah functional)

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Outline

ROF Denoising

- State the denoising problem as minimizing $J(u)$ under the constraints

$$\int_\Omega u\,dx = \int_\Omega u_o\,dx, \quad \int_\Omega (u-u_0)^2\,dx = |\Omega|\sigma^2 \quad (|\Omega| = \text{area/volume of } \Omega)$$

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- Chambolle-Lions: there exists λ such the solution minimizes

$$E_{TV}(u) = \frac{1}{2}\int_\Omega (Ku - u_0)^2\,dx + \lambda \int_\Omega |\nabla u|\,dx$$

- Euler-Lagrange equation:

$$K^*(Ku - u_0) - \lambda \text{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0.$$

- The term $\text{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ is highly non linear. Problems especially when $|\nabla u| = 0$.
- In fact $\frac{\nabla u/}{|\nabla u|}(x)$ is the unit normal of the level line of u at x and $\text{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ is the (mean)curvature of the level line: not defined when the level line is singular or does not exist!

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Acar-Vogel

- Replace it by regularized version

$$|\nabla u|_\beta = \sqrt{|\nabla u|^2 + \beta}, \quad \beta > 0$$

- Acar - Vogel show that

$$\lim_{\beta \to 0} \left(J_\beta(u) = \int_\Omega |\nabla u|_\beta \, dx \right) = J(u).$$

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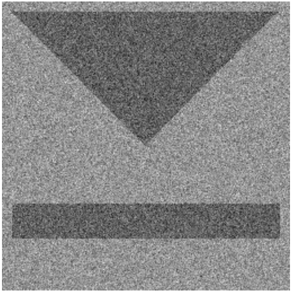
Example

Implementation by finite differences, fixed-point strategy, linearization.

Original



$\lambda = 1.5$, $\beta = 10^{-4}$

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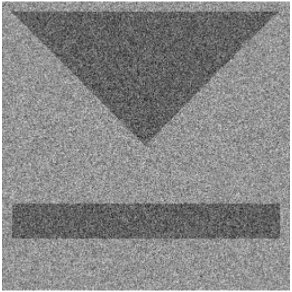

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Outline

- Filling u in the subset $H \subset \Omega$ where data is missing, denoise known data
- Inpainting energy (Chan & Shen):

$$E_{ITV}(u) = \frac{1}{2}\int_{\Omega \setminus H} (u - u_0)^2\, dx + \lambda \int_{\Omega} |\nabla u|\, dx$$

- Euler-Lagrange Equation:

$$(u - u_0)\chi - \lambda \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0.$$

($\chi(x) = 1$ is $x \notin H$, 0 otherwise).

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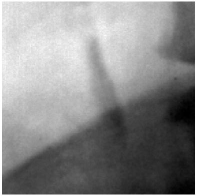
Degraded

Inpainted

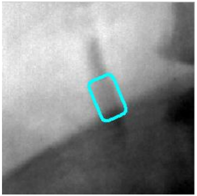
Segmention

Inpainting - driven segmention (Lauze, Nielsen 2008, IJCV)

Aortic calcifiction



Detection



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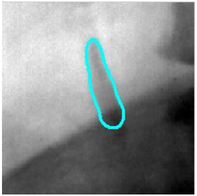

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$$J(u) = \int_\Omega |\nabla u| \, dx$$

u ***must have (weak) derivatives***.

- But we just saw that the computation is possible for a step-edge $u(x) = 0$, $x < 0$, $u(x) = a$, $x > 0$:

$$\int_{-1}^{1} |u'(x)| \, dx = |a|$$

- Can we avoid the use of derivatives of u?

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$$|\nabla u| = \nabla u \cdot \frac{\nabla u}{|\nabla u|}$$

(except when $\nabla u = 0$) and $\frac{\nabla u}{|\nabla u|}$ is the normal to the level lines of u, it has everywhere norm 1.

- Let V the set of vector fields $v(x)$ on Ω with $|v(x)| \leq 1$. I claim

$$J(u) = \sup_{v \in V} \int_\Omega \nabla u(x) \cdot v(x)\, dx$$

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Step-edge

- u the step-edge function defined in previous slides. We compute $J(u)$ with the new definition.
- here $W = \{\phi : [-1, 1] \to \mathbb{R}$ differentiable, $\phi(-1) = \phi(1) = 0, |\phi(x)| \leq 1\}$,

$$J(u) = \sup_{\phi \in W} \int_{-1}^{1} u(x)\phi'(x)\,dx$$

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$$\begin{aligned}\int_{-1}^{1} u(x)\phi'(x)\,dx &= a \int_{0}^{1} \phi'(x)\,dx \\ &= a\left(\phi(1) - \phi(0)\right) \\ &= -a\phi(0)\end{aligned}$$

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2D example

- B open set with regular boundary curve $partialB$, Ω large enough to contain B and χ_B the characteristic function of B

$$\chi_B(x) = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$$

- For $v \in W$, by the divergence theorem on B and its boundary ∂B

$$\begin{aligned} \int_\Omega \chi(x) \operatorname{div} v(x)\, dx &= \int_B \operatorname{div} v(x)\, dx \\ &= -\int_{\partial B} v(s) \cdot n(s)\, ds \end{aligned}$$

($n(s)$ is the exterior normal to ∂B)

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Sets of finite perimeter

- Let $H \subset \Omega$. If its characteristic function χ_H satisfies

$$J(\chi_H) < \infty$$

  H is called set of finite perimeter (and $Per_\Omega(H) := J(\chi_H)$ is its perimeter)

- This is used for instance in the Chan and Vese algorithm.
- If $J(u) < \infty$ and $H_t = \{x \in \Omega,\ u(x) < t\}$ the lower t-level set of u,

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Outline

Chambolle algorithm

- Let $K \in L^2(\Omega)$ the closure of the set $\{\operatorname{div} v,\ v \in C_0^1(\Omega)^2,\ |v(x)| \leq 1\}$ i.e. the image of W by div.
- Then
$$J(u) = \sup_{\phi \in K} \left(\int_\Omega u\,\phi\,dx = \langle u, \phi \rangle_{L^2(\Omega)} \right)$$
- Solution of the denoising problem arg.min $\int_\Omega (u - u_0)^2 + \lambda J(u)$ given by
$$u = u_0 - \pi_{\lambda K}(u_0)$$
with $\pi_{\lambda K}$ orthogonal projection onto the convex set λK (Chambolle).
- Needs a bit of convex analysis to show that: subdifferentials and subgradients, Fenchel transforms, indicators/characteristic functions and elementary results on them

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Fenchel Transform

- X Hilbert space, $f : X \to \mathbb{R}$ convex, proper. Fenchel transform of F:

$$F^*(v) = \sup_{u \in X} \left(\langle u, v \rangle_X - F(u) \right)$$

- Geometric meaning: take u^* such that $F^*(u^*) < +\infty$: the affine function

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- Interesting properties:
  - Convex
  - If Φ is the transform of F and $\lambda > 0$, then the transform of $u \mapsto \lambda F(\lambda^{-1}(u))$ is $\lambda\Phi$.
  - If F 1-homogeneous, i.e. $F(\lambda u) = \lambda F(u)$ then $F^*(u)$ only take values 0 and $+\infty$ as the property above implies $F^* = \lambda F^*$, $\lambda > 0$.
  - In that case, the set where $F^* = 0$ i a closed convex set of X, $F^* = \delta_C$, the indicator function of C,

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- subdifferential of F at u: $\partial F(u) = \{v \in X, F(w) - F(u) \geq \langle w - u, v \rangle, \forall w \in X\}$. $v \in \partial F(u)$ is a subgradient of F at u.
- Three fundamental (and easy) properties:
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$$\frac{u_0}{\lambda} \in \frac{u_0-u}{\lambda} + \frac{1}{\lambda}\partial J^*(\frac{u_0-u}{\lambda})$$

- Set $w = \frac{u_0-u}{\lambda}$: w satisfies

$$0 \in w - \frac{u_0}{\lambda} + \frac{1}{\lambda}\partial J^*(w)$$

- This is the subdifferential of the convex function

$$\frac{1}{2}\|w - u_0/\lambda\|^2 + \frac{1}{\lambda}J^*(w)$$

- But $J^*(w) = \delta_K(w)$: we get $w = \pi_K(g\lambda)$.

TV-denoising

- To minimize:

$$\frac{1}{2}\|u-u_0\|^2_{L^2(\Omega} + \lambda J(u)$$

- optimality condition:

$$0 \in u - u_0 + \lambda \partial J(u) \Leftrightarrow \frac{u_0 - u}{\lambda} \in \partial J(u)$$

- Duality

$$\frac{u_0}{\lambda} \in \frac{u_0 - u}{\lambda} + \frac{1}{\lambda}\partial J^*(\frac{u_0 - u}{\lambda})$$

- Set $w = \frac{u_0 - u}{\lambda}$: w satisfies

$$0 \in w - \frac{u_0}{\lambda} + \frac{1}{\lambda}\partial J^*(w)$$

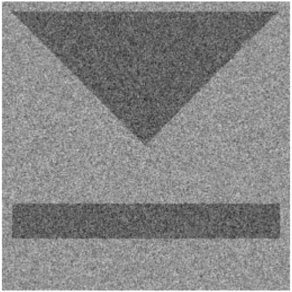
- This is the subdifferential of the convex function

$$\frac{1}{2}\|w - u_0/\lambda\|^2 + \frac{1}{\lambda}J^*(w)$$

- But $J^*(w) = \delta_K(w)$: we get $w = \pi_K(g\lambda)$.

Example

The usual original



Denoised by projection

Outline

Camerman Example

Solution of denoising energy present numerically stair-casing effect (Nikolova)

Original

xc

$\lambda = 100$

$\lambda = 500$

The gradient becomes “sparse”.

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The End