Key definitions, techniques & theorems so far:

① Definition: Multiplication of two matrices.

Let $A \in M_{mn}$, $B \in M_{np}$. Then

$$AB \in M_{mp}$$

is the matrix with entries

$$[AB]_{ij} = \sum_{k=1}^{n} [A]_{ik} [B]_{kj} \quad \text{for } \begin{array}{l} 1 \le i \le m \\ 1 \le j \le p. \end{array}$$

② Definition: Multiplication of a matrix and a vector.

Let $A \in M_{mn}$, $\vec{v} \in \mathbb{R}^n$. Then

$$A\vec{v} \in \mathbb{R}^m$$

is the vector with entries

$$[A\vec{v}]_i = \sum_{j=1}^{n} [A]_{ij} [\vec{v}]_j.$$

③ Theorem: Let $A = [\vec{u}_1 | \vec{u}_2 | \vec{u}_3 | \cdots | \vec{u}_n]$. In other words, $\vec{u}_i$ are the columns of the matrix A. Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. Then

$$A\vec{x} = x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots + x_n\vec{u}_n.$$

④ Theorem: Let $A \in M_{mn}$, $B \in M_{np}$, with

$$B = [\vec{u}_1 | \cdots | \vec{u}_p].$$

Then $AB = \left[A\vec{u}_1 \middle| \cdots \middle| A\vec{u}_p \right]$.

(5) Definition: Let $\vec{u}_1, \dots, \vec{u}_n \in \mathbb{R}^m$. A <u>linear</u> <u>combination</u> of $\vec{u}_i$ is a sum

$\alpha_1 \vec{u}_1 + \dots + \alpha_n \vec{u}_n$ for some choice of $\alpha_1, \dots, \alpha_n \in \mathbb{R}$.

(6) Definition: The <u>span</u> of $\vec{u}_1, \dots, \vec{u}_n \in \mathbb{R}^m$ is the set of all linear combinations of $\vec{u}_1, \dots, \vec{u}_n$.

We write the span as $\langle \vec{u}_1, \dots, \vec{u}_n \rangle \subset \mathbb{R}^m$.

(7) By theorem (3), $\langle \vec{u}_1, \dots, \vec{u}_n \rangle \subset \mathbb{R}^m$ equals the set of vectors of the form $A\vec{\alpha}$ where $A = \left[\vec{u}_1 \middle| \cdots \middle| \vec{u}_n \right]$ and $\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$.

(8) By (7) $\vec{v} \in \langle \vec{u}_1, \dots, \vec{u}_n \rangle$ is equivalent to the existence of $\vec{x} \in \mathbb{R}^n$ such that

$$A\vec{x} = \vec{v} \quad \text{where } A = \left[\vec{u}_1 \middle| \cdots \middle| \vec{u}_n \right].$$

This can be checked by row reduction.

(9) Let $A \in M \ldots \vec{b} \in \mathbb{R}^m$ ...

Let $A \in M_{mn}$, $\vec{b} \in \mathbb{R}^m$. We want to find $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{b}$.

Let $[A|\vec{b}]$ be the augmented matrix.

Perform row operations on $[A|\vec{b}]$ to bring it to row-reduced echelon form:

$$[A \mid \vec{b}] \xrightarrow{\text{RREF}} [B \mid \vec{c}].$$

If the i^{th} row of B is zero but $[\vec{c}]_i \neq 0$, then there are <u>no solutions</u>. We say the equation is <u>inconsistent</u>.

Suppose $[\vec{c}]_i = 0$ for each zero row of B. Then there is at least one solution $\vec{x}$, and we say the equation is <u>consistent</u>.

Let r = # of pivots of B.
If $r = n$, there is a unique solution.

If $r < n$, there are infinitely many solutions depending on $n-r$ free variables.

You should be able to write these down by solving the system of equations associated to

$B \cdot \vec{x} = \vec{c}$. Since B is RREF, this

is much easier than $A\vec{x} = \vec{b}$.

(9) Definition: Let $A \in M_{mn}$. The nullspace $N(A) \subset \mathbb{R}^n$ is the set of vectors $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = 0_m$.

(10) Let $\vec{x}_0$ be a solution of $A\vec{x} = \vec{b}$. Then the full solution set is given by

$$\{\vec{x}_0 + \vec{v} \mid \vec{v} \in N(A)\}.$$

We call $\vec{x}_0$ a "particular solution" and $\vec{v}$ a "homogenous solution".

(11) A set of vectors $\vec{u}_1, \dots, \vec{u}_n \in \mathbb{R}^m$ is called linearly independent if

$\alpha_1\vec{u}_1 + \dots + \alpha_n\vec{u}_n = 0_m$ implies $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

(12) Theorem: $\vec{u}_1, \dots, \vec{u}_n \in \mathbb{R}^m$ are linearly independent $\iff$ the nullspace of $A = [\vec{u}_1 | \cdots | \vec{u}_n]$ equals $\{0_n\}$.

(13) Definition: Let $A \in M_{n\times n}$. We say A is invertible if there exists $B \in M_{n\times n}$ such that $AB = BA = I_n$. We call B the <u>inverse</u> of A and write $B = A^{-1}$.

(14) Theorem: An invertible matrix has a <u>unique</u> inverse.

In other words: If
$$BA = AB = I_n$$
$$\& \; CA = AC = I_n$$
then $B = C$.

Moreover: if $BA = I_n$ then $AB = I_n$
& if $AB = I_n$ then $BA = I_n$.

(15) The following are equivalent:

a) $\vec{u}_1, \dots, \vec{u}_n \in \mathbb{R}^n$ are linearly independent

b) $A = \left[\vec{u}_1 \middle| \cdots \middle| \vec{u}_n\right]$ is non-singular.

c) $A^t = \begin{bmatrix} \vec{u}_1^{\,t} \\ \hline \vdots \\ \hline \vec{u}_n^{\,t} \end{bmatrix}$ is non-singular

d) A is invertible.

(16) Let $A \in M_{n\times n}$. Consider the augmented matrix $[A \mid I_n]$. Let's perform row-reduction on this augmented matrix.

RREF

Case 1: $[A \mid I_n] \rightsquigarrow [I_n \mid B]$.

Then A is invertible & $B = A^{-1}$.

Case 2: $[A \mid I_n] \overset{RREF}{\rightsquigarrow} [A' \mid B]$

where A' has a row of zeros.

Then A is not invertible.

(17) Theorem: Let $A, B \in M_{n \times n}$.

Then AB is invertible if and only if A & B are invertible. In this case, $(AB)^{-1} = B^{-1}A^{-1}$.

(18) Consider the system of linear equations

$$A\vec{x} = \vec{b}$$

& suppose A is a square invertible matrix. Then multiplying both sides on the left by A^{-1}, we find

$$\vec{x} = A^{-1}\vec{b}.$$

(19) Definition: the determinant of a square matrix $A \in M_{n \times n}$ is defined inductively by

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} [A]_{1i} \det(A(1|i))$$

where $A(1|i)$ is obtained from A by removing the 1^{st} row & i^{th} column.

(20) The above is called 'expanding along the 1^{st} row'. You can also expand along the j^{th} row or k^{th} column:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} [A]_{ji} \det(A(j|i))$$
$$= \sum_{i=1}^{n} (-1)^{i+k} [A]_{ik} \det(A(i|k)).$$

Some determinants are easier to compute: if A is upper triangular ($[A]_{ij} = 0$ for $i > j$) or lower triangular ($[A]_{ij} = 0$ for $i < j$)

Then $\det A = \prod_{i=1}^{n} [A]_{ii}$.

(21) If A & B differ by a row operation, the determinant changes in a simple way:

$$A \xrightarrow{R_i + \alpha R_j} B \qquad \det(A) = \det(B)$$

$$A \xrightarrow{R_i \leftrightarrow R_j} B \qquad \det(A) = -\det(B).$$

$$A \xrightarrow{\alpha R_i} B \qquad \alpha \det(A) = \det(B).$$

(22) Theorem: $\det(AB) = \det(A)\det(B)$.

for $A, B \in M_{n\times n}$.

Careful: if $\alpha \in \mathbb{R}$, $\det(\alpha A) = \alpha^n \det(A)$.

This is compatible with the previous theorem, since $\alpha A = (\alpha I_n) A$

& $\det(\alpha I_n) = \alpha^n$.

(23) Theorem: Let $A \in M_{n\times n}$.

$$\det(A) \neq 0 \iff A \text{ is non-singular.}$$

(24) In particular, if A has a zero row/column, or repeated rows/columns, or more generally a linear dependence among its rows or columns, then $\det(A) = 0$.

(25) Definition: Let $A \in M_{n\times n}$

(25) Definition: Let $A \in M_{n\times n}$.

Let $\vec{x} \in \mathbb{R}^n$. Then $\vec{x}$ is an <u>eigenvector</u> of A with <u>eigenvalue</u> $\lambda \in \mathbb{R}$

if
1) $\vec{x} \neq \vec{0}_n$.
2) $A\vec{x} = \lambda \vec{x}$.

(26) Let $\mathcal{E}_A(\lambda) \subset \mathbb{R}^n$ be the set of <u>all</u> $\vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \lambda \vec{x}$.

Thus it consists of eigenvectors + the zero vector.

(27) Theorem: $\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$.

(28) $\mathcal{E}_A(\lambda) \neq \{\vec{0}\}$ if and only if

$\det(A - \lambda I_n) = 0$.

(29) Definition: Let $p_A(x) = \det(A - x I_n)$.

We call it the characteristic polynomial of A.

(30) Theorem: Let $A \in M_{n\times n}$. Then $p_A(x)$ is a polynomial of degree n in x.

(31) Definition: Let $A, B \in M_{n\times n}$. We say A is similar to B if there exists $S \in M_{n\times n}$ invertible, s.t. $A = S^{-1} B S$.

(32) Theorem: Similarity is an equivalence relation:

1) A is similar to itself.

2) A similar to $B \iff B$ similar to A.

3) A similar to B, B similar to $C \Rightarrow A$ similar to C.

(33) Definition: A is <u>diagonalisable</u> if it is similar to a <u>diagonal</u> matrix D ($[D]_{ij} = 0$ for $i \neq j$).

(34) conceptual version
Theorem: Let $A \in M_{n\times n}$. A is diagonalisable if and only if there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

(34) computational version
Theorem. Let $A \in M_{n\times n}$. A is diagonalisable if and only if

$$\dim \mathcal{E}_A(\lambda_1) + \ldots + \dim \mathcal{E}_A(\lambda_r) = n$$

where $\lambda_1, \dots, \lambda_r$ are the roots of $p_A(x)$.

Note that you can compute $\dim \mathcal{E}_A(\lambda)$ for any λ, since $\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$ & you know how to compute dim of a nullspace via row-reduction.

(35) Let $\vec{x}_1, \dots, \vec{x}_n$ be a basis of $\mathbb{R}^n$ consisting of eigenvectors of $A \in M_{n\times n}$ with eigenvalues $\lambda_1, \dots, \lambda_n$.

Let $S = \left[\vec{x}_1 \middle| \cdots \middle| \vec{x}_n\right]$. Then

$$S^{-1}AS = D$$

where D is diagonal with $[D]_{ii} = \lambda_i$.

(36) Let A be diagonalisable. Then we can compute A^N very efficiently.

Let $A = S^{-1}DS$ where $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$.

Then $A^N = (S^{-1}DS)^N = S^{-1}D^N S$

$$= S^{-1}\begin{bmatrix} \lambda_1^N & & \\ & \ddots & \\ & & \lambda_n^N \end{bmatrix} S.$$

(37) Definition: let $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Then $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} [\vec{x}]_i [\vec{y}]_i$.

(38) 1) $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$.

2) $\langle \alpha \vec{x}_1 + \beta \vec{x}_2, \vec{y} \rangle = \alpha \langle \vec{x}_1, \vec{y} \rangle + \beta \langle \vec{x}_2, \vec{y} \rangle$.

3) $\langle \vec{x}, \vec{x} \rangle > 0$ for $\vec{x} \neq \vec{0}_n$.

(39) Definition: $\{\vec{x}_1, \dots, \vec{x}_r\} \subset \mathbb{R}^n$ is "orthogonal" if $\langle \vec{x}_i, \vec{x}_j \rangle = 0$ for $i \neq j$ & $\vec{x}_i \neq \vec{0}_n$ for all i.

(40) Definition: $\{\vec{x}_1, \dots, \vec{x}_r\} \subset V$ is an <u>orthogonal basis</u> of V if it is both orthogonal & a basis, i.e.:

1) $\langle \vec{x}_1, \dots, \vec{x}_r \rangle = V$

2) $\vec{x}_1, \dots, \vec{x}_r$ are linearly independent.

3) $\vec{x}_1, \dots, \vec{x}_r$ are orthogonal.

(Actually, 3) implies 2)).

41 Theorem: Let $\vec{x}_1, \ldots, \vec{x}_r$ be orthogonal.

Then

$$\langle \alpha_1 \vec{x}_1 + \ldots + \alpha_r \vec{x}_r, \beta_1 \vec{x}_1 + \ldots + \beta_r \vec{x}_r \rangle = \sum_{i=1}^{r} \alpha_i \beta_i \langle \vec{x}_i, \vec{x}_i \rangle.$$

42 Theorem: Let $\vec{x}_1, \ldots, \vec{x}_r$ be orthogonal, & let $\vec{u} = \sum_{i=1}^{r} \alpha_i \vec{x}_i$.

Then $\alpha_i = \dfrac{\langle \vec{x}_i, \vec{u} \rangle}{\langle \vec{x}_i, \vec{x}_i \rangle}$.

43 Theorem: Let $V \subset \mathbb{R}^n$ be a subspace.

Then there exists an orthonormal basis of V. (use Gram-Schmidt).