Recall: $A, B \in M_{n \times n}$

"A is similar to B" if

$$A = S^{-1} B S \quad \text{for } S \text{ an invertible}$$

matrix.

Theorem: Similarity is an equivalence relation:

1) A is similar to A. (reflexivity)

2) A is similar to $B \Rightarrow B$ similar to A. (symmetry)

3) A is similar to B, B similar to C

$\Rightarrow A$ similar to C. (transitivity)

Proof: 1) Need to find invertible $S \in M_{n\times n}$

such that $A = S^{-1}AS$.

Pick $S = I_n$. Then $S^{-1} = I_n$.

$$S^{-1}AS = I_n A I_n = A \checkmark.$$

2) Let $A = S^{-1}BS$. Need to find

S_2 such that $B = S_2^{-1}AS_2$.

Pick $S_2 = S^{-1}$. Then $S_2^{-1} = (S^{-1})^{-1} = S$.

If $A = S^{-1}BS$

multiplying on both sides by S on the left we get

$$SA = S(S^{-1}BS) = (SS^{-1})BS$$
$$= BS.$$

multiplying on the right by S^{-1}:

$$SAS^{-1} = (BS)S^{-1} = B(SS^{-1})$$
$$= B.$$

$$\Rightarrow \quad SAS^{-1} = S_2^{-1}AS_2 = B \checkmark.$$

3) Exercise.

Theorem: let A & B be similar matrices.

Then $p_A(x) = p_B(x)$.

Proof: • $p_A(x) = \det(A - xI_n)$.

• $p_B(x) = \det(B - xI_n)$.

• There exists $S \in M_{n\times n}$ s.t. $A = S^{-1}BS$.

$$p_A(x) = \det(S^{-1}BS - xI_n)$$
$$= \det(S^{-1}BS - x\underline{S^{-1}S})$$
$$= \det(S^{-1}(BS - xS))$$
$$= \det(S^{-1}(B - xI_n)S)$$
$$= \underline{\det(S^{-1})}\det((B - xI_n)S)$$
$$= \det(S^{-1})\det(B - xI_n)\underline{\det(S)}$$
$$= \det(S)^{-1}\det(B - xI_n)\det(S)$$
$$= \det(B - xI_n) = p_B(x).$$

Corollary: If A & B are similar, they have the same eigenvalues.

Proof: The eigenvalues of A (resp. B) are the roots of $p_A(x)$ (resp. $p_B(x)$).

Example of two matrices A, B which have same char polynomial but are <u>not</u> similar:

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Suppose A & B are similar. Then there exists S such that $B = S^{-1}AS$.

But $S^{-1}AS = S^{-1}I_2S = S^{-1}S = I_2$.

Contradiction! Hence A & B are <u>not</u> similar.

$$p_A(x) = \det(A - xI_2) = \det\begin{bmatrix} 1-x & 0 \\ 0 & 1-x \end{bmatrix} = (1-x)^2.$$

$$p_B(x) = \det(B - xI_2) = \det\begin{bmatrix} 1-x & 1 \\ 0 & 1-x \end{bmatrix} = (1-x)^2.$$

So $p_A(x) = p_B(x)$.

Diagonal matrices & diagonalizability

Definition: A matrix $A \in M_{n\times n}$ is diagonal if

$$[A]_{ij} = 0 \quad \text{for } i \neq j.$$

Example: $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Definition: $B \in M_{n \times n}$ is "<u>diagonalizable</u>" if it is <u>similar</u> to a diagonal matrix.

Example: Let $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Claim: there exists S such that $S^{-1}BS$ is diagonal.

Theorem: Let $A = \begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \lambda_3 & & & \\ & & & \lambda_4 & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{bmatrix}$ be diagonal with

$$[A]_{ii} = \lambda_i.$$

Then the eigenvalues of A are $\lambda_1, \dots, \lambda_n$.

Proof: $p_A(x) = \det(A - xI_n) = \det \begin{bmatrix} \lambda_1 - x & & \\ & \lambda_2 - x & \\ & & \ddots \end{bmatrix}$

$$\begin{bmatrix} \ddots & & & \\ & \lambda_3 - x & & \\ & & \ddots & \\ & & & \lambda_n - x \end{bmatrix}$$

$$= (\lambda_1 - x)(\lambda_2 - x) \cdots (\lambda_n - x)$$

The roots of this polynomial are $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Exercise: find $\mathcal{E}_A(\lambda_i)$.

Corollary: Let A be similar to a diagonal matrix D. The diagonal entries of D are the eigenvalues of A.

How to diagonalize a matrix: (if possible)

Theorem 1: A matrix $A \in M_{n \times n}$ is diagonalizable if & only if there is a basis of $\mathbb{R}^n$ consisting of eigenvectors for A.

Theorem 2: Let $\vec{x}_1, \ldots, \vec{x}_n$ be a basis of $\mathbb{R}^n$ consisting of eigenvectors for A.
Let $S = \begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{bmatrix}$.

Then $S^{-1}AS$ is diagonal.

Example: Let $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

1) $\mathbb{R}^2$ has a basis of eigenvectors of B.

$$p_B(x) = \det \begin{bmatrix} 1-x & 1 \\ 0 & -x \end{bmatrix} = (1-x)(-x).$$

Roots of $p_B(x)$: $\lambda_1 = 1$, $\lambda_2 = 0$.

$$\mathcal{E}_B(1) = \mathcal{N}(B - 1 \cdot I_2) = \mathcal{N}\left(\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\right)$$

$$= \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$$

$$\mathcal{E}_B(0) = \mathcal{N}(B - 0 \cdot I_2) = \mathcal{N}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

$$= \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle.$$

The eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ form a basis of $\mathbb{R}^2$.

By theorem 1, B is diagonalizable.

By theorem 2; if $S = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$,

$S^{-1}BS$ is diagonal.

Check: $S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$.

$$S^{-1}BS = \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

diagonal.

Theorem: Suppose $A \in M_{n\times n}$ with eigenvalues $\lambda_1, \dots, \lambda_k$

Theorem: Suppose A is $n \times n$ with eigenvalues $\lambda_1, \dots, \lambda_r$.

Let $\vec{x}_1 \cdots \vec{x}_{k_1}$ be a basis of $\mathcal{E}_A(\lambda_1)$.

Let $\vec{y}_1, \dots, \vec{y}_{k_2}$ be a basis of $\mathcal{E}_A(\lambda_2)$.

$\vdots$

Let $\vec{z}_1, \dots, \vec{z}_{k_r}$ be a basis of $\mathcal{E}_A(\lambda_r)$.

Then A is diagonalizable $\iff$ there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A

$$\iff \boxed{k_1 + k_2 + \dots + k_r = n.}$$

A basis for $\mathbb{R}^n$ is given by $\vec{x}_1, \dots, \vec{x}_{k_1}, \dots\dots, \vec{z}_1, \dots, \vec{z}_{k_r}$.

Remark: k_i is the dimension of $\mathcal{E}_A(\lambda_i) = \mathcal{N}(A - \lambda_i I_n)$.

Theorem: Let $A \in M_{n \times n}$ with n distinct eigenvalues.

Then A is diagonalizable.

Example: $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$$\det(A - xI_2)$$

$$= \det \begin{bmatrix} 1-x & 2 \\ 2 & 3-x \end{bmatrix}$$

$$= (1-x)(3-x) - 4.$$

$$= x^2 - 4x + 3 - 4$$

$$= x^2 - 4x - 1.$$

Roots of $p_A(x) = \dfrac{4 \mp \sqrt{16+4}}{2} = \dfrac{4 \mp \sqrt{20}}{2}$

2 distinct eigenvalues $\Rightarrow$ A is diagonalizable.

Application of diagonalizability:

Question: Let A be a diagonalisable matrix.

Compute A^{1000}.

Answer: Write $A = S^{-1}BS$ where B is diagonal.

$$B = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

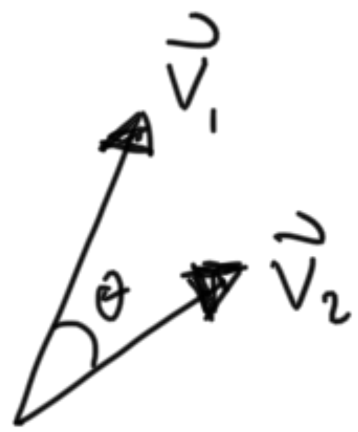
$$\begin{aligned} A^{1000} &= A \cdot A \cdot A \cdots\cdots A \\ &= S^{-1}B\underbrace{S \cdot S^{-1}}B\underbrace{S \cdot S^{-1}}B\underbrace{S \cdots}\cdots \underbrace{S^{-1}}BS \\ &= S^{-1}B \cdot B \cdot B \cdots\cdots BS \\ &= S^{-1}B^{1000}S. \end{aligned}$$

Now we only need to compute.

Exercise: show $B^{1000} = \begin{bmatrix} \lambda_1^{1000} & & & \\ & \lambda_2^{1000} & & \\ & & \ddots & \\ & & & \lambda_n^{1000} \end{bmatrix}$.

Hence $A^{1000} = S^{-1} \begin{bmatrix} \lambda_1^{1000} & & & \\ & \lambda_2^{1000} & & \\ & & \ddots & \\ & & & \lambda_n^{1000} \end{bmatrix} S$.

Inner products



$\vec{v}_1 \cdot \vec{v}_2 = \cos\theta \, |\vec{v}_1| |\vec{v}_2|$.

Definition: Let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$.

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \sum_{i=1}^{n} [\vec{v}_1]_i [\vec{v}_2]_i.$$

Ex: Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

$$\langle \vec{v}_1, \vec{v}_2 \rangle = 3\cdot 0 + 1\cdot(-1) + 2\cdot 2 = 1.$$

Theorem: 1) $\langle \vec{v}_1 + \vec{v}_2, \vec{u} \rangle = \langle \vec{v}_1, \vec{u} \rangle + \langle \vec{v}_2, \vec{u} \rangle.$

2) $\langle \alpha \vec{v}, \vec{u} \rangle = \alpha \langle \vec{v}, \vec{u} \rangle$

3) $\langle \vec{v}, \vec{u} \rangle = \langle \vec{u}, \vec{v} \rangle$

4) $\langle \vec{v}, \vec{v} \rangle > 0$ if $\vec{v} \neq \vec{0}.$

Proof: 1) $\langle \vec{v}_1 + \vec{v}_2, \vec{u} \rangle = \sum_{i=1}^{n} [\vec{v}_1 + \vec{v}_2]_i [\vec{u}]_i$

$$= \sum_{i=1}^{n} ([\vec{v}_1]_i + [\vec{v}_2]_i) [\vec{u}]_i$$

$$= \sum_{i=1}^{n} [\vec{v}_1]_i [\vec{u}]_i + [\vec{v}_2][\vec{u}]_i$$

$$= \langle \vec{v}_1, \vec{u} \rangle + \langle \vec{v}_2, \vec{u} \rangle. \checkmark.$$

4) $\langle \vec{v}, \vec{v} \rangle = \sum_{i=1}^{n} [\vec{v}]_i [\vec{v}]_i$

$$= \sum_{i=1}^{n} [\vec{v}]_i^2.$$

Since $[\vec{v}]_i^2 \geq 0$, this is a sum of non-negative numbers. Thus it vanishes if and only if $[\vec{v}]_i^2 = 0$ for $i = 1, \ldots, n$. Hence $[\vec{v}]_i = 0$ for $i = 1, \ldots, n$, hence $\vec{v} = \vec{0}_n$.

Theorem: 1) $\langle \alpha \vec{v} + \beta \vec{u}, \vec{w} \rangle = \alpha \langle \vec{v}, \vec{w} \rangle + \beta \langle \vec{u}, \vec{w} \rangle$

2) $\langle \vec{w}, \alpha \vec{v} + \beta \vec{u} \rangle = \alpha \langle \vec{w}, \vec{v} \rangle + \beta \langle \vec{w}, \vec{u} \rangle.$

3) $\langle \vec{0}, \vec{u} \rangle = \langle \vec{u}, \vec{0} \rangle = 0.$

4) If $\langle \vec{x}, \vec{v} \rangle = 0$ for all $\vec{x} \in \mathbb{R}^n$, then

$\vec{v} = \vec{0}.$

5) If $\langle \vec{v}, \vec{x} \rangle = \langle \vec{w}, \vec{x} \rangle$ for all $\vec{x} \in \mathbb{R}^n$

then $\vec{v} = \vec{w}.$ (Exercise: prove this).

Definition: (generalization of the length of a vector $\vec{x} \in \mathbb{R}^3$). Let $\vec{v} \in \mathbb{R}^n$.

Define $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}.$

Remark: we are allowed to take the square root because

$\langle \vec{v}, \vec{v} \rangle \geq 0$ for all $\vec{v} \in \mathbb{R}^n$.

Example: Let $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

$$= \sqrt{1 \cdot 1 + 0 \cdot 0 + (-1)(-1) + (-1)(-1)}$$

$$= \sqrt{3}$$

Definition: We say $\vec{v}$ is a <u>unit</u> vector if $\|\vec{v}\| = 1$.

Theorem: Let $\vec{v} \in \mathbb{R}^n$, $\vec{v} \neq \vec{0}$.

Then $\left(\frac{1}{\|\vec{v}\|}\right) \cdot \vec{v}$ is a unit vector.

scalar vector.

Proof: $$\left\langle \frac{1}{\|\vec{v}\|} \vec{v}, \frac{1}{\|\vec{v}\|} \vec{v} \right\rangle = \frac{1}{\|\vec{v}\|} \left\langle \vec{v}, \frac{1}{\|\vec{v}\|} \vec{v} \right\rangle$$

$$= \frac{1}{\|\vec{v}\|^2} \langle \vec{v}, \vec{v} \rangle$$

$$= \frac{1}{\left(\sqrt{\langle \vec{v}, \vec{v} \rangle}\right)^2} \langle \vec{v}, \vec{v} \rangle$$

$$= \frac{1}{\langle \vec{v}, \vec{v} \rangle} \langle \vec{v}, \vec{v} \rangle$$

$$= 1. \quad \checkmark.$$

Definition: An orthogonal Set $S \subset \mathbb{R}^n$ is a set of vectors $S = \{\vec{x}_1, \dots, \vec{x}_k\}$ such that

$\langle \vec{x}_i, \vec{x}_j \rangle = 0$ for $i \neq j$.

& $\vec{x}_i \neq 0$ for all i.

Theorem: Fix an orthogonal set $\{\vec{x}_1, \dots, \vec{x}_k\}$.

Let $\vec{v} = a_1\vec{x}_1 + \dots + a_k\vec{x}_k$, Let $\vec{w} = \beta_1 \vec{x}_1 + \dots + \beta_k \vec{x}_k$.

Then $\langle \vec{v}, \vec{w} \rangle = \alpha_1 \beta_1 \langle \vec{x}_1, \vec{x}_1 \rangle + \alpha_2 \beta_2 \langle \vec{x}_2, \vec{x}_2 \rangle$

$+ \dots$

$+ \alpha_k \beta_k \langle \vec{x}_k, \vec{x}_k \rangle$.

Proof: $\langle \vec{v}, \vec{w} \rangle = \left\langle \sum_{i=1}^{k} \alpha_i \vec{x}_i, \sum_{j=1}^{k} \beta_j \vec{x}_j \right\rangle$

$= \sum_{i,j=1}^{k} \langle \alpha_i \vec{x}_i, \beta_j \vec{x}_j \rangle$

$= \sum_{i,j=1}^{k} \alpha_i \langle \vec{x}_i, \beta_j \vec{x}_j \rangle$

$= \sum_{i,j=1}^{k} \alpha_i \beta_j \langle \vec{x}_i, \vec{x}_j \rangle$

$$= \sum_{i,j=1} \alpha_i \beta_j \langle \vec{x}_i, \vec{x}_j \rangle$$ This is zero if $i \neq j$

$$= \sum_{i=j=1}^{k} \alpha_i \beta_i \langle \vec{x}_i, \vec{x}_i \rangle .$$