Eigenvalues & eigenvectors

Recall: Let A be a size n square matrix.

We call $\vec{x} \in \mathbb{R}^n$ an <u>eigenvector</u> of A with <u>eigenvalue</u> $\lambda \in \mathbb{R}$ if

1) $\vec{x} \neq \vec{0}_n$
2) $A\vec{x} = \lambda \vec{x}$.

Def: $\mathcal{E}_A(\lambda) \subset \mathbb{R}^n$ is the set of <u>all</u> $\vec{x}$ s.t.

$$A\vec{x} = \lambda \vec{x}.$$

Thm: $\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$.

Def: Let A be a square size n matrix. We say $\lambda \in \mathbb{R}$ is an <u>eigenvalue</u> of A if there exists an eigenvector $\vec{x} \in \mathbb{R}^n$ with eigenvalue λ.

Question: how to calculate the eigenvalues & eigenspaces of A?

$$\begin{aligned}
\lambda \text{ is eigenvalue of } A &\iff \mathcal{E}_A(\lambda) \text{ contains nonzero vector} \\
&\iff \mathcal{N}(A - \lambda I_n) \text{ contains nonzero vector} \\
&\iff A - \lambda I_n \quad \text{singular.} \\
&\iff \det(A - \lambda I_n) = 0.
\end{aligned}$$

Def: Let A be a square size n matrix.

The "characteristic polynomial" of A is the polynomial

$$p_A(x) = \det(A - xI_n).$$

Thm: λ is an eigenvalue of $A \iff \lambda$ is a root of $p_A(x)$ $(p_A(\lambda) = 0)$.

Example: Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

1) Compute $p_A(x)$.

2) Compute eigenvalues of A.

3) Compute $\mathcal{E}_A(\lambda)$ for each eigenvalue of A.

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1) $$p_A(x) = \det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 2-x & 1 \\ 1 & 2-x \end{bmatrix}\right)$$

$$= (2-x)^2 - 1 = x^2 - 4x + 3.$$

2) The eigenvalues of A are the roots of $p_A(x)$.

Use quadratic formula:

$$p_A(\lambda) = 0 \iff \lambda = \frac{4 \pm \sqrt{16-12}}{2}$$

used that $p_A(x)$ is a deg 2 polynomial.

$$= \frac{4 \pm \sqrt{4}}{2}$$

This won't always be true.

$$= \frac{4 \pm 2}{2}$$

Hence the eigenvalues of A are

$$\lambda_1 = \frac{4-2}{2} = 1. \qquad \lambda_2 = \frac{4+2}{2} = 3.$$

3) Compute $\mathcal{E}_A(1)$ & $\mathcal{E}_A(3)$.

(this means: find a basis of each subspace).

We have

$$\mathcal{E}_A(1) = \mathcal{N}(A - 1 \cdot I_2)$$

$$= \mathcal{N}\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \mathcal{N}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right).$$

A basis for $\mathcal{N}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$ is given by

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \qquad \mathcal{N}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle.$$

$$\begin{aligned}
\mathcal{E}_A(3) &= \mathcal{N}(A - 3I_2) \\
&= \mathcal{N}\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= \mathcal{N}\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right).
\end{aligned}$$

A basis for $\mathcal{N}\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right)$ is given by

by $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $\qquad \mathcal{N}\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle.$

We conclude: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has eigenvalues $1, 3$

& eigenspaces $\mathcal{E}_A(1) = \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle$

$$\mathcal{E}_A(3) = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle.$$

In general, to compute the eigenspaces of a square matrix A:

1) Compute $p_A(x) = \det(A - xI_n)$

   if you can compute determinants, you can do this.

2) Find the roots of $p_A(x)$.

   This can be very difficult / impossible.

3) Given roots $\lambda_1, \lambda_2, \dots, \lambda_r$ of $p_A(x)$,

   calculate bases of $\mathcal{E}_A(\lambda_i) = \mathcal{N}(A - \lambda_i I_n)$, using techniques from this class.

4) $\mathcal{E}_A(\lambda_i) = \langle \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{m_i} \rangle$

basis of $\mathcal{E}_A(\lambda_i)$

Ex: $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 1 & 0 & 0 \end{bmatrix}$

$$p_A(x) = \det(A - xI_3)$$

$$= \det\left(\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 1 & 0 & 0 \end{bmatrix} - x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} 1-x & 2 & 0 \\ 0 & 3-x & 4 \\ 1 & 0 & -x \end{bmatrix} \right)$$

expand along 1^{st} column

$$= (1-x)\,\det\begin{bmatrix}3-x & 4\\ 0 & -x\end{bmatrix}$$

$$- \;0\cdot\det(\cdot\!-)$$

$$+ \;1\cdot\det\begin{bmatrix}2 & 0\\ 3-x & 4\end{bmatrix}$$

$$= (1-x)\big((3-x)(-x)-4\cdot 0\big)$$

$$+ \;1\cdot\big((2\cdot 4)-(3-x)\cdot 0\big)$$

$$= (1-x)(3-x)(-x)+8$$

$$= -x^3+4x^2-3x+8.$$

finding roots of this may be difficult.

Theorem: Let A be a square matrix of size n.

Then $p_A(x)$ is a polynomial in x of degree n.

Proof: we skip this.

Similarity of matrices

Definition: Let A & B be square matrices of size n. We say A & B are similar if there exists an invertible matrix S of size n s.t.

$$\boxed{A = S^{-1}BS}$$

We say A is similar to B via S,

and we call $S^{-1}BS$ a similarity transformation.

Thm: Similarity is an equivalence relation, i.e.

1) It is reflexive: A is similar to A.
2) It is symmetric: A is similar to $B \iff B$ is similar to A.
3) It is transitive: If A is similar to B
   & B is similar to C
   then A is similar to C.