Determinants: Cramer's rule & formula for A^{-1}.

Recall: Given $A \in M_{n \times n}$, $\det(A)$ is a real number such that $\det(A) \neq 0$ exactly if A is non-singular. We can compute in various ways (expand along rows or columns...).

Recall: If $A \rightsquigarrow B$ by a row operation, the determinant of B is obtained from $\det(A)$ by a simple rule: Eg $A \xrightarrow{R_i \leftrightarrow R_j} B$, $\det(B) = -\det(A)$.

$A \rightsquigarrow B \quad \det(B) = \alpha \det(A).$

Theorem: Let $A, B \in M_{n \times n}$.

$$\text{Then } \det(AB) = \det(A)\det(B).$$

To prove this, you can use the behaviour of det under row operations.

Example: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix}.$$

$$\det(A) = ad - bc \qquad \det(B) = xw - yz$$

$$\det(AB) = (ax + bz)(cy + dw) - (ay + bw)(cx + dz).$$

Exercise: check $\det(AB) = \det(A)\det(B)$.

Can use $\det(AB) = \det(A)\det(B)$ to reprove

$$\boxed{\det(A) \neq 0 \iff A \text{ non-singular.}}$$

Indeed, suppose A is nonsingular. Then there exists $B \in M_{n \times n}$ such that $BA = I_n$.

Then $\det(B)\det(A) = \det(I_n) = 1$

If $\det(A) = 0$, we get a contradiction. So $\det(A) \neq 0$.

Conversely, suppose $\det(A) \neq 0$. We saw last class how to show that A is non-singular.

Theorem: Let A be an invertible matrix with inverse A^{-1}. Then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Proof: $A \cdot A^{-1} = I_n$.

Hence $\det(A A^{-1}) = \det(I_n) = 1.$

But $\det(A \cdot A^{-1}) = \det(A) \det(A^{-1})$

But $\det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1})$.

So $\det(A) \cdot \det(A^{-1}) = 1$. $\square$.

Example: $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

$$A \cdot A^{-1} = \begin{bmatrix} 1\cdot 1 + 2\cdot 0 & 1\cdot(-2) + 2\cdot 1 \\ 0\cdot 1 + 1\cdot 0 & 0(-2) + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$\det(A) = 1\cdot 1 - 2\cdot 0 = 1$.

$\det(A^{-1}) = 1\cdot 1 - (-2)\cdot 0 = 1$ ✓.

Theorem: Cramer's rule. Let $A \in M_{n\times n}$ be invertible.

Let $\underline{A\vec{x} = \vec{b}}$ where $\vec{x}, \vec{b} \in \mathbb{R}^n$.

Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \end{bmatrix}$ Then

Let $x = \begin{bmatrix} \vdots \\ x_n \end{bmatrix}$. Then

$$x_k = \frac{\det(M_k)}{\det(A)}$$

where $M_k = \begin{bmatrix} \vec{A}_1 & \cdots & \vec{b} & \cdots & \vec{A}_n \end{bmatrix}$ is obtained from

k^{th} column

A by replacing the k^{th} column by $\vec{b}$.

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

① Check that A is invertible $\Leftrightarrow \det(A) \neq 0$.

$$\det(A) = 1\cdot 3 - 2\cdot 0 = 3. \quad \checkmark$$

(2) Apply Cramer's rule to compute $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ s.t. $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$.

$$x_1 = \frac{\det(M_1)}{\det(A)} \quad \text{where} \quad M_1 = \left[\vec{b} \,\middle|\, \vec{A}_2\right] = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\det(M_1) = 1\cdot 3 - 2\cdot 1 = 3 - 2 = 1.$$
$$\det(A) = 1\cdot 3 - 2\cdot 0 = 3.$$

$$x_1 = 1/3.$$

$x_2 = \det(M_2)$ where $M_2 = [\vec{A}_1 | \vec{b}]$

$$x_2 = \frac{\det(M_2)}{\det(A)} \quad \text{where} \quad M_2 = \left[A_1 \middle| b \right]$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\det(M_2) = 1 \cdot 1 - 1 \cdot 0 = 1$$
$$\det(A) = 3.$$

$$x_2 = 1/3.$$

Hence our solution is $\vec{x} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$.

Remark: Cramer's rule allows you to compute a single entry of the solution $\vec{x}$ of $A\vec{x} = \vec{b}$ without also computing all the other entries.

Proof of Cramer's rule:

Consider the matrix $\boxed{X_k}$ obtained from I_n by replacing the k^{th} column by $\vec{x}$.

Ex: $$X_k = \begin{bmatrix} 1 & 0 & 0 & x_1 & 0 \\ 0 & 1 & 0 & x_2 & 0 \\ 0 & 0 & 1 & x_3 & 0 \\ 0 & 0 & 0 & \boxed{x_4} & 0 \\ 0 & 0 & 0 & x_5 & 1 \end{bmatrix} = \left[\vec{e}_1 \,\middle|\, \vec{e}_2 \,\middle|\, \vec{e}_3 \,\middle|\, \vec{x} \,\middle|\, \vec{e}_5\right].$$

Then $$A X_k = \left[A\vec{e}_1 \,\middle|\, A\vec{e}_2 \,\middle|\, \cdots \,\middle|\, \underset{k^{th}\text{ column}}{A\vec{x}} \,\middle|\, \cdots \,\middle|\, A\vec{e}_n\right]$$

$$= \left[\vec{A}_1 \,\middle|\, \vec{A}_2 \,\middle|\, \cdots \,\middle|\, \vec{b} \,\middle|\, \cdots \,\middle|\, \vec{A}_n\right] = \boxed{M_k}.$$

where we used that $A\vec{e}_i = \vec{A}_i$

& $A\vec{x} = \vec{b}$.

So $AX_k = M_k$. It follows that

$$\boxed{\det(A)\det(X_k) = \det(AX_k) = \det(M_k).}$$

Exercise: Show that $\det(X_k) = x_k$.

Hint: expand along the k^{th} row.

It follows that $\det(A) \cdot x_k = \det(M_k)$

$$\implies x_k = \frac{\det(M_k)}{\det(A)}. \qquad \square$$

Theorem: Let $A \in M_{n \times n}$ be invertible.

Then $$\boxed{[A^{-1}]_{ji} = (-1)^{i+j} \frac{\det(A(i|j))}{\det(A)}}$$

Notice the similarity to Cramer's rule.

Example: Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$. Want to know $[A^{-1}]_{13}$.

We have $$[A^{-1}]_{13} = (-1)^{1+3} \cdot \frac{\det(A(3|1))}{\det(A)}$$

$$= (-1)^{1+3} \frac{\det\left(\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\right)}{}$$

$$\det(A)$$

(used that A is upper triangular, so $\det(A)$ is the product of diagonal entries)

$$= \frac{(-1)^4 \cdot (2\cdot 3 - 1\cdot 1)}{(3\cdot 1\cdot 2)}$$

$$= \frac{1\cdot(6-1)}{6} = \frac{5}{6}.$$

Proof: Uses Cramer's rule. Sketch:

$$A \cdot A^{-1} = I_n.$$

Hence $A \cdot A^{-1}_i = \vec{e}_i$.

Cramer's rule tells us: $[A^{-1}_i]_j = \dfrac{\det(M_j)}{\det(A)}$

where $M_j = \left[\vec{A}_1 \middle| \dots \middle| \vec{e}_i \middle| \dots \vec{A}_n \right]$

$\uparrow$ j^{th} column.

Check that $\det(M_j) = \det(A(i|j)) \cdot (-1)^{i+j}$.

Eigenvalues & eigenvectors

__Definition:__ Let A be a square matrix of size n.
Let $\vec{x} \in \mathbb{R}^n$ be a nonzero vector.

We say $\vec{x}$ is an __eigenvector__ of A with __eigenvalue__ $\lambda \in \mathbb{R}$

if $\boxed{A\vec{x} = \lambda \vec{x}.}$

Example: Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$A\vec{u} = \begin{bmatrix} 2\cdot 1 + 1\cdot 1 + 1\cdot 1 \\ 1\cdot 1 + 2\cdot 1 + 1\cdot 1 \\ 1\cdot 1 + 1\cdot 1 + 2\cdot 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = 4\cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 4\vec{u}.$$

So $\vec{u}$ is an <u>eigenvector</u> of A with <u>eigenvalue</u> 4.

Let $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Then $A\vec{v} = \begin{bmatrix} 2\cdot 1 + 1\cdot(-1) + 1\cdot 0 \\ 1\cdot 1 + 2(-1) + 1\cdot 0 \\ 1\cdot 1 + 1(-1) + 2\cdot 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1\cdot \vec{v}$,

So $\vec{v}$ is an <u>eigenvector</u> of A with <u>eigenvalue</u> $\boxed{\lambda = 1.}$

Theorem: Let $\vec{x}$ be an eigenvector of A with eigenvalue λ. $(A\vec{x} = \lambda\vec{x})$.

Then given $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\alpha\vec{x}$ is <u>also</u> an eigenvector of A with eigenvalue λ.

Proof: $A(\alpha\vec{x}) = \alpha(A\vec{x}) = \alpha(\lambda\vec{x}) = (\alpha\lambda)\vec{x}$
$= \lambda(\alpha\vec{x})$.

So $A(\alpha\vec{x}) = \lambda(\alpha\vec{x})$. $\square$.

Theorem: Suppose $\vec{v}, \vec{w} \in \mathbb{R}^n$ are eigenvectors of $A \in M_{n\times n}$ with eigenvalue λ. $(A\vec{v} = \lambda\vec{v},\ A\vec{w} = \lambda\vec{w})$.

Then $\vec{v}+\vec{w}$ is <u>also</u> an eigenvector of A with eigenvalue λ.

<u>Proof</u>: Want to check that $A(\vec{v}+\vec{w}) = \lambda(\vec{v}+\vec{w})$.

But $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$

$= \lambda\vec{v} + \lambda\vec{w}$ (use that $\vec{v}, \vec{w}$ are eigenvectors with eigenvalue λ)

$= \lambda(\vec{v}+\vec{w})$. $\square$

Conclusion of the two previous theorems: we can make new eigenvectors from old by adding / multiplying by scalars.

Definition: The eigenspace of A with eigenvalue λ

is the subset of vectors $\vec{x} \in \mathbb{R}^n$
such that $A\vec{x} = \lambda\vec{x}$.

Theorem (rephrasing of 2 previous theorems):
If $\vec{x}, \vec{y}$ belong to the eigenspace of A w/eigenvalue λ, then so does $\alpha\vec{x}$, $\alpha \in \mathbb{R}$, $\vec{x}+\vec{y}$.

In other words: $\boxed{\text{The eigenspace is a } \underline{\text{subspace}} \text{ of } \mathbb{R}^n.}$

Remark: our definition of eigenvector does not include the zero vector $\vec{0}_n$, even though $A\vec{0}_n = \lambda \cdot \vec{0}_n$

for all $\lambda \in \mathbb{R}$. So the eigenspace with eigenvalue λ consists of all eigenvectors with eigenvalue λ + the zero vector.

We write $\mathcal{E}_A(\lambda) \subset \mathbb{R}^n$ for the eigenspace of A with eigenvalue $\lambda \in \mathbb{R}$.

<u>How do we find eigenvectors??</u>

Fix $\lambda \in \mathbb{R}$. Try to find an eigenvector $\bar{x}$ with eigenvalue λ. We thus want a solution to

$$\boxed{A\bar{x} = \lambda \bar{x}}$$

$$\boxed{A\vec{x} = \lambda \vec{x}.}$$

We can rewrite this as $A\vec{x} - \lambda\vec{x} = \vec{0}_n$.

We rewrite $\lambda\vec{x}$ as $\lambda \cdot I_n(\vec{x})$.

$$A\vec{x} - \lambda I_n(\vec{x}) = \vec{0}_n.$$

We group terms: $(A - \lambda I_n)\vec{x} = \vec{0}_n$.

We conclude:

Theorem: $\vec{x}$ is an eigenvector of A with eigenvalue λ

$\Rightarrow \vec{x} \in N(A - \lambda I_n)$.

Conversely, if $\vec{x} \in N(A - \lambda I_n)$ & $\vec{x} \neq \vec{0}_n$,

$\vec{x}$ is an eigenvector, with eigenvalue λ.

Example: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $\lambda = 1$.

Find an eigenvector $\vec{x}$ with eigenvalue $\lambda = 1$.

$$\mathcal{N}(A - 1 \cdot I_2) = \mathcal{N}\left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$
$$= \mathcal{N}\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right).$$

Any nonzero vector $\vec{x} \in \mathcal{N}\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right)$ will do.

$\mathcal{N}\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \text{ where } a \in \mathbb{R} \right\}$.

So $\vec{x} = \begin{bmatrix} a \\ 0 \end{bmatrix}$ is an eigenvector of A with eigenvalue 1 for any $a \in \mathbb{R}$.

We can summarize the above by:

Theorem: Let A be a square matrix of size n.

Let $\lambda \in \mathbb{R}$.

$$\boxed{\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n).}$$