Matrix inverses, part 2

Recall: Let $A \in M_{nn}$ be a square matrix.
We say A is <u>invertible</u> if there exists $B \in M_{nn}$ such that

$$AB = I_n \quad \& \quad BA = I_n.$$

We showed that
- A invertible $\Rightarrow$ A non-singular.
- (*) A non-singular $\Rightarrow$ $\exists B \in M_{nn}$ such that $AB = I_n$.

(we used row reduction on $[A \mid I_n] \xrightarrow{\text{RREF}} [I_n \mid B]$)

In order to conclude that A non-singular $\Rightarrow$ A invertible, we still need to show that $BA = I_n$.

We will prove this in a few steps. The goal is to show that if $AB = I_n$ then

$$BA = I_n.$$

Theorem: A is nonsingular $\Rightarrow A^t$ is non-singular.

Proof: Recall that if $A \overset{RREF}{\rightsquigarrow} B$,

then a basis of $\mathcal{R}(A) = \mathcal{C}(A^t)$

is given by the nonzero columns of B^t

(= nonzero rows of B). Since A is

nonsingular, $B = I_n$, hence $\mathcal{C}(A^t) = \mathbb{R}^n$

$\Rightarrow A^t$ is non-singular.

Corollary: If A is nonsingular, then $\exists C \in M_{nn}$

such that $CA^t = I_n$

Proof: Use the previous theorem & $(*)$.

Take the transpose of both sides of the equation

$$CA^t = I_n$$

to get

$$(CA^t)^t = I_n^t$$

$$\Rightarrow (A^t)^t C^t = I_n \qquad (\text{use } I_n^t = I_n)$$

$$\Rightarrow AC^t = I_n$$

Hence if A is non-singular, $\exists B, C \in M_{nn}$ such that $BA = I_n = AC^t$.

We now show that $B = C^t$. Indeed:

$$BA = I_n$$

$$\Rightarrow (BA)C^t = I_n C^t$$ (multiply both sides by C^t)

$$\Rightarrow B(AC^t) = I_n C^t$$ (associativity of matrix multiplication)

$$\Rightarrow B(I_n) = C^t$$

$$\Rightarrow B = C^t.$$

Conclusion: A non-singular $\Leftrightarrow \exists B \in M_{nn}$ such that $BA = AB = I_n$.

(i.e. A is <u>invertible</u> with inverse B).

Corollary: If $C, D \in M_{nn}$, both non-singular, then CD is also non-singular.

Proof: C non-singular $\Rightarrow$ C has an inverse C^{-1}.

D " $\Rightarrow$ D " " " D^{-1}.

$$(D^{-1}C^{-1})\cdot CD = D^{-1}(C^{-1}C)D$$
$$= D^{-1}(I_n)D$$
$$= I_n.$$

Hence CD is invertible $\Rightarrow$ CD is non-singular.

Theorem: Let $CA = DA = I_n$. Then $C = D$.

(in other words, A has a unique inverse).

Proof: let $B \in M_{nn}$ s.t. $AB = I_n$.

$$CA = DA$$

$$\Rightarrow (CA)B = (DA)B \qquad \text{(multiply on the right by } B\text{).}$$

$$\Rightarrow C(AB) = D(AB)$$

$$\Rightarrow C I_n = D I_n$$

$$\Rightarrow C = D. \quad \square$$

Note: uniqueness of the inverse is what allows us to write $\underline{A^{-1}}$ for the inverse of A.

Exercise: Let A be

Exercise: Let A be a square invertible matrix.

Prove

1) $(A^t)^{-1} = (A^{-1})^t$.

2) $(\alpha A)^{-1} = \alpha^{-1} A^{-1}$

3) In general: $A^{-1} + B^{-1} \neq (A+B)^{-1}$

(in fact, A invertible & B invertible does not imply $A+B$ is invertible).

Answer: 1) Want to show: $(A^{-1})^t \cdot A^t = I_n$.

We know

$$A A^{-1} = I_n.$$

Taking transpose on both sides:

$$(AA^{-1})^t = I_n^t = I_n.$$

$$\Rightarrow (A^{-1})^t \cdot A^t = I_n. \quad \square$$

2) Want to show: $(\alpha^{-1}A^{-1}) \cdot (\alpha A) = I_n$.

Use commutativity of scalar multiplication:

$$(\alpha^{-1} A^{-1})(\alpha A)$$

$$= A^{-1}(\alpha^{-1}\alpha) A$$

$$= A^{-1} A = I_n. \quad \square$$

3) We need to find a counterexample

to $A^{-1} + B^{-1} = (A+B)^{-1}$.

We pick $A, B \in M_{11}$ (one-by-one

matrices, i.e. real numbers).

$$A = [1] \qquad B = [2].$$

$$A^{-1} = [1] \qquad B^{-1} = [2^{-1}] = [1/2].$$

$$A + B = [1] + [2] = [3].$$

$$(A+B)^{-1} = [3^{-1}] = [1/3]$$

$$A^{-1} + B^{-1} = [1] + [1/2] = [3/2] \neq [1/3].$$

Key properties of the inverse: Let $A \in M_{nn}$.

1) If A is invertible, it has a unique inverse.

2) $BA = I_n \iff AB = I_n$.

3) A is non-singular if and only if it is invertible.

4) $(AB)^{-1} = B^{-1}A^{-1}$.

Exercise: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- Show that A is invertible if and only if $ad - bc \neq 0$.
- If $ad - bc \neq 0$, then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$.

Hint: show that if $ad - bc = 0$, $A \xrightarrow{RREF} B$ where B has a zero row.