Rank, nullity & matrix inverses

Recall: Given a subspace $V \subset \mathbb{R}^n$, the <u>dimension</u> of V, $\dim(V)$, is the # of vectors in a basis of V.

Let $A \in M_{mn}$. The rank of A, $r(A)$, is the dimension of the column space:

$$r(A) = \dim \mathcal{C}(A).$$

The nullity of A, $n(A)$, is the dimension of the nullspace:

$$n(A) = \dim \mathcal{N}(A)$$

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Calculating $r(A)$ & $n(A)$:

Theorem: Let $A \in M_{mn}$. $A \xrightarrow{RREF} B$.

Then ① $r(A) = \#$ of pivots of B
$= \#$ nonzero rows of B.

② $n(A) = \#$ columns $- r(A)$
$= n - r(A)$

not the same

Proof: ① As we saw last class, there is a basis of $\mathcal{C}(A)$ indexed by the pivot columns of B.

② As we saw, there is a basis of $\mathcal{N}(A)$

indexed by the pivotless columns of B.

Example: $A = \begin{bmatrix} 3 & 3 & 6 \\ 2 & 2 & 4 \\ 2 & 1 & 3 \end{bmatrix}$ Compute $r(A)$, $n(A)$.

$$A \rightsquigarrow \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 2 & 1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = B$$

pivots

$r(A) = 2$, $n(A) = 3 - 2 = 1$.

Corollary: Let $A \in M_{mn}$.

"rank-nullity theorem"

$$\boxed{\underbrace{r(A)}_{\text{rank}} + \underbrace{n(A)}_{\text{nullity}} = \underbrace{n}_{\#\text{ columns}}}$$

Proof: By the previous theorem,

$$n(A) = n - r(A).$$

Hence
$$r(A) + n(A) = r(A) + (n - r(A))$$
$$= n. \quad \square$$

Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \end{bmatrix}$ $\qquad r(A) =$

$n(A) =$

$$\begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$A \xrightarrow{RREF} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad r(A) = 1$$

$$n(A) = 2 - 1 = 1.$$

Example: $A = \begin{bmatrix} 1 & 3 & 5 & 1 \\ 2 & 4 & 2 & 1 \end{bmatrix}$, $\quad r(A) = 2$

$n(A) = 4 - 2 = 2.$

Theorem: Let $A \in M_{nn}$ be a square matrix.

Then the following are equivalent:

a) A is non-singular.

b) $r(A) = n$

c) $n(A) = 0$.

Proof: We show that a) implies b) & c).

$$\text{a)} \iff \mathcal{N}(A) = \{0_n\} \Rightarrow n(A) = 0. \quad (c).$$

$$\Rightarrow r(A) = n - n(A)$$

$$= n - 0$$

$$= n. \quad (b).$$

We leave the implication c) $\Rightarrow$ a) as an exercise.

We now have many different equivalent characterizations of nonsingular matrices:

Theorem: Let $A \in M_{nn}$. TFAE (the following are equivalent):

1) A is non-singular.

2) $A \xrightarrow{RREF} I_n$.

3) $\mathcal{N}(A) = \{0_n\}$.

4) $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ for all $\vec{b} \in \mathbb{R}^n$.

5) The columns of A are linearly independent,

6) A is <u>invertible</u> (more about this later)

7) $\mathcal{C}(A) = \mathbb{R}^n$.

8) The columns of A are a basis of $\mathbb{R}^n$

8) The columns of A are a basis of $\mathbb{R}^n$.

9) $r(A) = n$

10) $n(A) = 0$.

Matrix inverses

Recall: $I_n \in M_{nn}$ is the square matrix

with $[I_n]_{ii} = 1$

$[I_n]_{ij} = 0$ for $i \neq j$.

Example: $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Theorem: Let $\vec{x} \in \mathbb{R}^n$. Then $I_n \vec{x} = \vec{x}$.

Proof:
$$[I_n \vec{x}]_j = \sum_{i=0}^{n} \underline{[I_n]_{ji}} [\vec{x}]_i$$

equals zero unless $j=i$

$$= [I_n]_{jj} [\vec{x}]_j$$

$$= 1 \cdot [\vec{x}]_j = [\vec{x}]_j \quad \square$$

Definition: Let A & B be square of size n, such that $AB = I_n$, $BA = I_n$. Then we say A is <u>invertible</u> and B is the <u>inverse</u> of A. We write $B = A^{-1}$.

Example: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$A \cdot B = \begin{bmatrix} 1\cdot 1 + 1\cdot 0 & 1\cdot(-1) + 1\cdot 1 \\ 0\cdot 1 + 1\cdot 0 & 0\cdot(-1) + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$B \cdot A = \begin{bmatrix} 1\cdot 1 - 1\cdot 0 & 1\cdot 1 - 1\cdot 1 \\ 0\cdot 1 + 1\cdot 0 & 0\cdot 1 + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

So B is the <u>inverse</u> of A, and A is <u>invertible</u>.

Suppose we want to solve the system of linear eq.

$$(*) \quad A\vec{x} = \vec{b}.$$

Suppose A is invertible, with inverse B.

Then we can multiply by B on both sides of $(*)$:

$$\underbrace{BA}\vec{x} = B \cdot \vec{b}$$

$$I_n \vec{x} = B\vec{b}$$

$$\vec{x} = \underbrace{B\vec{b}}_{\text{solution.}}$$

Not all square matrices are invertible. Exercise: find a 2×2 matrix which is <u>not</u> invertible. Find one that is.

Example of a non-invertible matrix:

let $A = \begin{bmatrix} 0 & 0 \end{bmatrix}$. Then for all $B \in M_{22}$,

$$A \cdot B = B \cdot A = 0 \quad (\text{the zero matrix})$$

$$\neq I_2.$$

More generally, any singular matrix is non-invertible:

Theorem: Let $A \in M_{n \times n}$. Then A is invertible if and only if A is non-singular.

To fully prove this will take a while.

One direction is easier:

Suppose A is invertible, with inverse A^{-1}.

Let $\vec{x} \in N(A) \iff A\vec{x} = \vec{0}_n$.

Then
$$A^{-1}(A\vec{x}) = A^{-1}\cdot\vec{0}_n$$
$$(A^{-1}A)\vec{x} = A^{-1}\cdot\vec{0}_n$$
$$\vec{x} = \vec{0}_n.$$

$\Rightarrow N(A) = \{0_n\}$.

$$\boxed{\text{This proves } A \text{ invertible} \Rightarrow A \text{ nonsingular.}}$$

Now, suppose A is non-singular. In particular,

$A \xrightarrow{RREF} I_n$. We will use this to produce it.

Recipe for A^{-1} from row-reduction:

Consider the augmented matrix $[A \mid \underline{I_n}]$.

Then if $A \xrightarrow{RREF} I_n$,

$$[A \mid I_n] \xrightarrow{RREF} [I_n \mid \underline{A^{-1}}].$$

Example: let $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

form augmented matrix $[A \mid I_2] = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1' = \frac{1}{2}R_1} \left[\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2' = R_2 - R_1} \left[\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & -1/2 & -1/2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2' = -2R_2} \left[\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1' = R_1 - \frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

Claim: $A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$

Check: $A \cdot \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot 0 + 1\cdot 1 & 2\cdot 1 + 1(-2) \\ 1\cdot 0 + 0\cdot 1 & 1\cdot 1 + 0(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Example: find inverse of $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

$\}\ R_1' = R_1 - 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow\ R_3' = \tfrac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$$

$$\Big\downarrow\ R_2' = R_2 - R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1/3 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$$

Claim: $A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1/3 \end{bmatrix}.$

Check: $\begin{bmatrix} 1 & -2 & 0 \\ \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ \end{bmatrix}$

Check: $$A^{-1} \cdot A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 1 - 2\cdot 0 + 0\cdot 0 & 1\cdot 2 - 2\cdot 1 + 0\cdot 0 & 1\cdot 2 - 2\cdot 1 + 0\cdot 3 \\ 0\cdot 1 + 1\cdot 0 - \frac{1}{3}\cdot 0 & 0\cdot 2 + 1\cdot 1 - \frac{1}{3}\cdot 0 & 0\cdot 2 + 1\cdot 1 - \frac{1}{3}\cdot 3 \\ 0\cdot 1 + 0\cdot 0 + \frac{1}{3}\cdot 0 & 0\cdot 2 + 0\cdot 1 + \frac{1}{3}\cdot 0 & 0\cdot 2 + 0\cdot 1 + \frac{1}{3}\cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

Why does this recipe work? We want to find B s.t. $AB = I_n$. If $B = [\vec{b}_1 | \cdots | \vec{b}_n]$ then

$$AB = [A\vec{b}_1 | \cdots | A\vec{b}_n].$$

So we want $A\vec{b}_1 = \vec{e}_1$, $A\vec{b}_2 = \vec{e}_2$, etc.
where $\vec{e}_i$ is the i^{th} column of I_n.
We can solve for $\vec{b}_i$ by doing row reduction

$$\left[\begin{array}{c|c} A & \vec{e}_i \end{array}\right].$$

Suppose $[A \mid \vec{e}_i] \xrightarrow{RREF} [I_n \mid \vec{d}_i]$.
Then solving, we find $\vec{b}_i = \vec{d}_i$.

Our recipe simply does row reduction for all the $\vec{b}_i$ at once:

$$\left[\begin{array}{c|c|c|c|c|c} A & \vec{c}_1 & \vec{c}_2 & \vec{c}_3 & \cdots & \vec{c}_n \end{array}\right] = \left[\begin{array}{c|c} A & I_n \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{c|c|c|c} I_n & \vec{d}_1 & \cdots & \vec{d}_n \end{array}\right]$$

$$= \left[\begin{array}{c|c} I_n & B \end{array}\right].$$

This shows that our recipe produces B such that $AB = I_n$. To see why $BA = I_n$, we will have to wait till next class.

Exercise: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Compute A^{-1}

using the above method.

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & \text{[illegible]} \end{bmatrix}.$$

We finish off today with an easier property of the inverse:

Theorem: Let A & B be invertible $n \times n$ matrices. Then $A \cdot B$ is <u>also</u> invertible

& $(AB)^{-1} = B^{-1} \cdot A^{-1}$

Note: opposite order!

Proof: Want to show $(B^{-1} \cdot A^{-1})(AB) = I_n$.

Use associativity of matrix multiplication:

$$
\begin{aligned}
& (B^{-1}A^{-1})(AB) \\
&= B^{-1}(A^{-1}A)B \\
&= B^{-1}(I_n)B \\
&= B^{-1}(I_n B) \\
&= B^{-1}B \\
&= I_n \checkmark
\end{aligned}
$$