Row spaces & more bases

Recall: If $A \in M_{mn}$ with columns $\vec{u}_1, \dots, \vec{u}_n$, then the column space of A, $\mathcal{C}(A) \subset \mathbb{R}^m$ is the span of $\vec{u}_1, \dots, \vec{u}_n$:

$$\mathcal{C}(A) = \langle \{ \vec{u}_1, \dots, \vec{u}_n \} \rangle.$$

If $\vec{v} \in \mathbb{R}^m$, then $\vec{v} \in \mathcal{C}(A)$ if and only if (1) $A\vec{x} = \vec{v}$ is consistent (i.e. there <u>exists</u> $\vec{x}$ such that (1) holds).

Column spaces can be understood as nullspaces:

Example: let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix}$

describe the col. space of A, $\mathcal{C}(A) \subseteq \mathbb{R}^4$.

equivalently, characterize all $\vec{v} \in \mathbb{R}^4$ s.t.

$A\vec{x} = \vec{v}$ is consistent.

augmented matrix $[A \mid \vec{v}] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & v_1 \\ 2 & 1 & 0 & v_2 \\ 3 & 0 & 1 & v_3 \\ 4 & 0 & 0 & v_4 \end{array}\right]$

$\downarrow \; R_2' = R_2 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & v_1 \\ 0 & -1 & 0 & v_2-2v_1 \\ 3 & 0 & 1 & v_3 \\ 4 & 0 & 0 & v_4 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & v_1 \\ 0 & -1 & 0 & v_2-2v_1 \\ 0 & -3 & 1 & v_3-3v_1 \\ 0 & -4 & 0 & v_4-4v_1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & v_1 \\ 0 & 1 & 0 & 2v_1-v_2 \\ 0 & -3 & 1 & v_3-3v_1 \\ 0 & -4 & 0 & v_4-4v_1 \end{array}\right]$$

$$\Big\downarrow \; R_3' = R_3 + 3R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & v_1 \end{array}\right.$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 2v_1-v_2 \\ 0 & 0 & 1 & v_3-3v_1 + 3(2v_1-v_2) \\ 0 & -4 & 0 & v_4-4v_1 \end{array}\right]$$

$$\Big\downarrow \quad R_4' = R_4 + 4R_2$$

$$\begin{array}{r} \\ \\ \\ \text{zero row} \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 0 & v_1 \\ 0 & 1 & 0 & 2v_1-v_2 \\ 0 & 0 & 1 & 3v_1-3v_2+v_3 \\ 0 & 0 & 0 & v_4-4v_1+4(2v_1-v_2) \end{array}\right]$$

$A\vec{x}=\vec{v}$ is consistent if and only if:

$$v_4-4v_1+4(2v_1-v_2)=0$$

$$v_4+4v_1-4v_2=0$$

$$\Rightarrow \vec{v}\in \mathcal{C}(A) \iff v_4+4v_1-4v_2=0$$

$$\iff \vec{v}\in N(D)$$

where $D = \begin{bmatrix} 4 & -4 & 0 & 1 \end{bmatrix}$

Exercise: let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$. Express $\mathcal{C}(A)$ as $\mathcal{N}(D)$ for some matrix D.

Answer:

$$\left[\begin{array}{cc|c} 1 & 2 & v_1 \\ 1 & 3 & v_2 \\ 1 & 1 & v_3 \\ 1 & 0 & v_4 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & v_1 \\ 0 & 1 & v_2 - v_1 \\ 0 & -1 & v_3 - v_1 \\ 0 & -2 & v_4 - v_1 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{r} \\ \\ \text{zero row} \rightarrow \\ \text{zero row} \rightarrow \end{array} \left[\begin{array}{cc|c} 1 & 2 & v_1 \\ 0 & 1 & v_2 - v_1 \\ 0 & 0 & v_3 - v_1 + v_2 - v_1 \\ 0 & 0 & v_4 - v_1 + 2(v_2 - v_1) \end{array}\right]$$

$A\vec{x} = \vec{v}$ is consistent if and only if

$$v_3 - v_1 + v_2 - v_1 = 0$$

$$\iff \boxed{-2v_1 + v_2 + v_3 = 0}$$

$$v_4 - v_1 + 2(v_2 - v_1) = 0$$

$$\iff \boxed{-3v_1 + 2v_2 + v_4 = 0}$$

So $A\vec{x} = \vec{v}$ is consistent if and only if

$$\vec{v} \in N(D) \qquad D = \begin{bmatrix} -2 & 1 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{bmatrix}$$

Theorem: Let $A \in M_{mn}$ matrix.

$A \xrightarrow{RREF} B$. Let B have z zero rows.

Then $C(A) = N(D)$ where D is a certain

$z \times m$ matrix.

Let $A \in M_{mn}$. Want a basis of $\mathcal{C}(A)$.

Eg: $A = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 3 \\ 3 & 0 & 0 & 1 & 1 \end{bmatrix}$, need to decide which columns to keep & which to throw away.

Theorem: let $A \xrightarrow{RREF} B$, and let B have pivot columns with indices $d_1, d_2, \dots, d_r$.

$\left(\text{Eg: } B = \begin{array}{cccccc} d_1 & & d_2 & & & d_3 \\ \end{array} \begin{bmatrix} \boxed{1} & * & 0 & * & * & 0 \\ 0 & 0 & \boxed{1} & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)$. Then a basis for $\mathcal{C}(A)$ is given by:

$$\vec{A}_{d_1}, \vec{A}_{d_2}, \dots, \vec{A}_{d_r}$$

where

$$A = \left[\vec{A}_1 \,\middle|\, \cdots \,\middle|\, \vec{A}_n\right].$$

Example: $A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ \begin{matrix} 1 & 2 & \end{matrix} \end{array}$ Find a basis of $\mathcal{C}(A)$.

$$A \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{array}{c} \begin{matrix} d_1=1 & d_2=2 & \end{matrix} \\ \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \end{bmatrix} \\ \text{RREF} \end{array} = B$$

$\Rightarrow$ basis of $\mathcal{C}(A)$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Exercise: let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ find a basis of $\mathcal{C}(A)$.

Answer: $A \overset{RREF}{\rightsquigarrow} \begin{bmatrix} \boxed{1} & 0 & \boxed{2} \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$d_1 = 1 \quad d_2 = 2$

$$\text{basis} = \{\vec{A}_1, \vec{A}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Note: $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Recall, if $A \in M_{n \times n}$, then A is non-singular if and only if $\vec{A}_1 \dots \vec{A}_n$ are a basis of $\mathbb{R}^n$.

Can deduce this from our theorem above:

A non-singular $\Leftrightarrow$ $A \rightsquigarrow \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} = I_n.$

$\Leftrightarrow$ all columns of RREF of A are pivots.

This shows that $\vec{A}_1, \dots, \vec{A}_n$ are a basis of $\mathcal{C}(A)$. Need to also show $\mathcal{C}(A) = \mathbb{R}^n$. (i.e. $A\vec{x} = \vec{v}$ is always consistent, for any $\vec{v} \in \mathbb{R}^n$)

Row space of a matrix.

Definition: Let $A \in M$ ... $A = \begin{bmatrix} \vec{w}_1^t \\ \hline \vec{w}_2^t \\ \vdots \end{bmatrix}$

Definition: Let $A \in M_{m \times n}$, $A = \begin{bmatrix} \vec{u}_1^t \\ \vdots \\ \vec{u}_m^t \end{bmatrix}$

then $R(A)$ (row space of A) is the span of $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m \in \mathbb{R}^n$.

i.e. $R(A) = \langle \{\vec{u}_1, \dots, \vec{u}_m\} \rangle$.

Can reformulate in terms of the column space of a different matrix: Alternative definition: Let $A \in M_{mn}$.

The row space $R(A) = \mathcal{C}(A^t)$.

Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$. $R(A) = \mathcal{C}(A^t) \subset \mathbb{R}^2$

$A^t = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$A^t \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \qquad d_1 = 1 \quad d_2 = 3.$$

$$\text{basis of } \mathcal{C}(A^t) = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \text{basis of } R(A).$$

Theorem: Suppose A & B are row-equivalent.
Then $R(A) = R(B)$.

Proof: Strategy of proof is to show that if B is obtained from A by a <u>single</u> row operation, then $R(A) = R(B)$. Then we can deduce that if B is obtained from A by multiple row operations, $R(A) = R(B)$.

Recall: row operations are: 1) $R_i \to \alpha R_i \quad \alpha \neq 0$

2) $R_i \to R_i + \alpha R_j$

3) $R_i \leftrightarrow R_j$.

Let $A = \begin{bmatrix} \vec{u}_1^{\,t} \\ \hline \vdots \\ \hline \vec{u}_m^{\,t} \end{bmatrix} \qquad A^t = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{bmatrix}$

Apply row operation of type 1) to A:

$A \xrightarrow{1)} B = \begin{bmatrix} \vec{u}_1^{\,t} \\ \hline \vdots \end{bmatrix} \qquad B^t = \Bigg[\; \Bigg]$

$$\begin{bmatrix} \vdots \\ \hline \alpha \vec{u}_i \\ \hline \vdots \\ \hline \vec{u}_m \end{bmatrix} \qquad \begin{bmatrix} \vec{u}_1 & \cdots & \alpha\vec{u}_i & \cdots & \vec{u}_m \end{bmatrix}$$

Need to show $R(A) = R(B) \iff \mathcal{C}(A^t) = \mathcal{C}(B^t)$.

col of A^t

Let $\vec{v} \in \mathcal{C}(A^t)$, i.e. $\vec{v} = a_1\vec{u}_1 + \dots + a_i\vec{u}_i + \dots + \vec{u}_m$

(use $\alpha \neq 0$) $\quad = a_1\vec{u}_1 + \dots + \left(\frac{a_i}{\alpha}\right)(\alpha\vec{u}_i) + \dots + \vec{u}_m.$

col of B^t

$\in \mathcal{C}(B^t)$.

Hence $R(A) \subset R(B)$.

Let $\vec{v} \in \mathcal{C}(B^t)$, i.e. $\vec{v} = a_1\vec{u}_1 + \dots + a_i(\alpha\vec{u}_i) + \dots + \vec{u}_m.$

$= a_1\vec{u}_1 + \dots + (a_i\alpha)\vec{u}_i + \dots + \vec{u}_m$

$\in \mathcal{C}(A^t)$.

Hence $R(B) \subset R(A)$

$\Rightarrow R(B) = R(A)$

We now check that operations of type 2 preserve row spaces.

Let $A = \begin{bmatrix} \vec{u}_1^{\,t} \\ \hline \vdots \\ \hline \vec{u}_m^{\,t} \end{bmatrix}$ $\quad A^t = \left[\begin{array}{c|c|c} \vec{u}_1 & \cdots & \vec{u}_m \end{array}\right]$

$A \xrightarrow{R_i' = R_i + \alpha R_j} B = \begin{bmatrix} \vec{u}_1^{\,t} \\ \hline \vdots \\ \hline \vec{u}_i^{\,t} + \alpha \vec{u}_j^{\,t} \\ \hline \vdots \end{bmatrix}$ $\quad B^t = \left[\begin{array}{c|c|c|c} \vec{u}_1 & \cdots & \vec{u}_i + \alpha \vec{u}_j & \cdots \end{array}\right]$

$R(A), R(B) \quad \mathcal{C}(A^t), \mathcal{C}(B^t)$

Want to show: $R(A) = R(B) \iff C(A^t) = C(B^t)$.

Suppose $\vec{v} \in \mathcal{C}(A^t)$; i.e. $\vec{v} = a_1\vec{u}_1 + \dots + a_m\vec{u}_m$.

$$\vec{v} = a_1\vec{u}_1 + \dots + a_j\vec{v}_j + \dots + a_i\vec{v}_i + \dots + a_m\vec{v}_m$$

$$= a_1\vec{u}_1 + \dots + a_j\vec{u}_j + \dots + a_i(\vec{u}_i + \alpha\vec{u}_j - \alpha\vec{u}_j) + \dots + a_m\vec{u}_m$$

$$= a_1\vec{u}_1 + \dots + \left[a_j\vec{u}_j - a_i\alpha\vec{u}_j\right] + \dots + a_i(\vec{u}_i + \alpha\vec{u}_j) + \dots + a_m\vec{u}_m$$

$$= a_1\vec{u}_1 + \dots + (a_j - a_i\alpha)\vec{u}_j + \dots + a_i(\vec{u}_i + \alpha\vec{u}_j) + \dots + a_m\vec{u}_m$$

$$\in \mathcal{C}(B^t)$$

Hence $\mathcal{C}(A^t) \subset \mathcal{C}(B^t)$.

Let $\vec{v} \in \mathcal{C}(B^t)$, $\vec{v} = a_1\vec{u}_1 + \dots + a_i(\vec{u}_i + \alpha\vec{u}_j) + \dots + a_m\vec{u}_m$

$$= a_1\vec{u}_1 + \dots + a_i\vec{u}_i + a_i\alpha\vec{u}_j + \dots + a_m\vec{u}_m$$

$$= a_1\vec{u}_1 + \dots + (a_j + a_i\alpha)\vec{u}_j + \dots + a_i\vec{u}_i + \dots + a_m\vec{u}_m$$

$$\in \mathcal{C}(A^t).$$

Hence $\mathcal{C}(B^t) \subset \mathcal{C}(A^t)$

Hence $\mathcal{C}(B^t) = \mathcal{C}(A^t)$.

Question: If A & B are row equivalent, do we have $\mathcal{C}(A) = \mathcal{C}(B)$?

Answer: No. Example: Let $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in M_{21}$

$B = \begin{bmatrix} 1 \end{bmatrix} \in M_{21}$.

$\begin{bmatrix} 0 \end{bmatrix}$

Then $A \xrightarrow{RREF} B$.

But $\mathcal{C}(A) = \left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\rangle \neq \mathcal{C}(B) = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$.

Basis of row space:

Theorem: Let $A \in M_{mn}$, $A \xrightarrow{RREF} B$.

Let S be the nonzero columns of B^t.

Then 1) $R(A) = \langle S \rangle$.

2) S is linearly independent.

(so S forms a basis of $R(A)$).

Example: Let $A = \begin{bmatrix} 1 & 1 \\ \ldots & \ldots \end{bmatrix}$

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1 \quad 0} \\ 0 \quad 0 \\ \boxed{0 \quad 1} \end{bmatrix} = B$$

$$B^t = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 0 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} 0 \\ 1 \end{matrix}} \end{bmatrix}$$

$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $R(A)$.

This produces a very <u>nice</u> basis of $R(A)$.

(basis vectors in S tend to have a lot of zeros).

We can use this theorem to give a new basis of $\mathcal{C}(A)$ for any matrix $A \in M_{mn}$.

Theorem: $\mathcal{C}(A) = \mathcal{R}(A^t)$. (Recall: $\mathcal{R}(A) = \mathcal{C}(A^t)$).

Proof: Let $B = A^t$.

$$\mathcal{R}(B) = \mathcal{C}(B^t) \quad \text{(by definition).}$$

But $B^t = (A^t)^t = A$.

So $\mathcal{R}(A^t) = \mathcal{C}(A)$ as desired.

So, having expressed $\mathcal{C}(A)$ as $\mathcal{R}(A^t)$, can use our basis of $\mathcal{R}(A^t)$ to get a basis of $\mathcal{C}(A)$.

Example: $A = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$ find basis of $C(A)$.

(equivalently, basis of $R(A^t)$).

$$A^t = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $R(A^t)$

$= C(A)$.