Nonsingular matrices & basis

Recall: A square matrix $A \in M_{n \times n}$ is called nonsingular if $\mathcal{N}(A) = \{0_n\}$.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is nonsingular

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is singular.

Theorem: A is nonsingular if and only if the columns of A are a basis of $\mathbb{R}^n$.

(i.e. if $A = [\vec{u}_1 | \dots | \vec{u}_n]$, $\vec{u}_1, \dots, \vec{u}_n$ are a basis).

Proof: Show A nonsingular $\Rightarrow \vec{u}_1 \dots \vec{u}_n$ is a basis

Need to show 1) $\langle \vec{u}_1, \dots, \vec{u}_n \rangle = \mathbb{R}^n$.

2) $\vec{u}_1, \dots, \vec{u}_n$ are linearly independent.

Let's start with 2):

Suppose $a_1\vec{u}_1 + \dots + a_n\vec{u}_n = 0_n$.

We have $a_1\vec{u}_1 + \dots + a_n\vec{u}_n = A \cdot \vec{x}$

where $\vec{x} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$.

So $\vec{x} \in N(A) \Rightarrow \vec{x} = 0_n$ by assumption

$\Rightarrow a_1 = a_2 = \dots = a_n = 0. \checkmark$

Now let's show 1).

We need to show that for <u>all</u> $\vec{v} \in \mathbb{R}^n$,

$\exists a_1 \dots a_n$ s.t. $a_1\vec{u}_1 + \dots + a_n\vec{u}_n = \vec{v}$.

$\Leftrightarrow \exists \vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{v}$. $\left(\vec{x} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right)$.

$A \xrightarrow{RREF} B \qquad \{\vec{0}\} = N(A) = N(B)$.

It follows that B has no zero rows.

$\Rightarrow B\vec{y} = \vec{v}$ always has a solution $\vec{y}$

$\Rightarrow A\vec{x} = \vec{v}$ always has a solution $\vec{x}$. $\checkmark$

We conclude that if A is nonsingular, then $\vec{u}_1 \dots \vec{u}_n$ are a basis of $\mathbb{R}^n$.

We leave the converse ($\vec{u}_1 \dots \vec{u}_n$ are basis $\Rightarrow A$ nonsingular)

as an exercise.

Consequence: Let $\vec{u}_1 \dots \vec{u}_n \in \mathbb{R}^n$. To check if $\vec{u}_1, \dots, \vec{u}_n$ is a basis, let

$A = [\vec{u}_1 | \dots | \vec{u}_n]$.

Perform row reduction $A \overset{RREF}{\rightsquigarrow} B$.

Check that B has no zero rows $/ = I_n$.

Example: Verify that $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

is a basis of $\mathbb{R}^3$.

Proof: Let $A = [\vec{u}_1 | \vec{u}_2 | \vec{u}_3] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

no zero rows ✓

Theorem: Suppose $A \in M_{n \times n}$ is non-singular.

Let $\vec{u}_1 \dots \vec{u}_n$ be a basis of $\mathbb{R}^n$

(not necessarily having any relation to A).

Then $A\vec{u}_1, \dots, A\vec{u}_n$ is <u>also</u> a basis of $\mathbb{R}^n$.

Proof: Consider $B = \left[A\vec{u}_1 \middle| \cdots \middle| A\vec{u}_n \right]$.

$A\vec{u}_1, \ldots, A\vec{u}_n$ is a basis if and only if

$N(B) = \{0_n\}$

Let $C = \left[\vec{u}_1 \middle| \cdots \middle| \vec{u}_n \right]$. Since $\vec{u}_1, \ldots, \vec{u}_n$ is a basis,

$N(C) = \{0_n\}$.

But, $B = A \cdot C$.

Since A & C are nonsingular, so is B.

It follows that $A\vec{u}_1, \ldots, A\vec{u}_n$ is <u>also</u> a

basis of $\mathbb{R}^n$.

Now suppose $A \in M_{m \times n}$ $\quad m < n$.

$$A = \begin{array}{c} n \\ \left[\begin{array}{c|c|c} \vec{u}_1 & \cdots\cdots & \vec{u}_n \end{array}\right] \end{array} m \, .$$

As we have seen previously, the columns $\vec{u}_1, \ldots \vec{u}_n$ of A are never a basis of $\mathbb{R}^m$.

Suppose instead $A \in M_{m \times n}$ $\quad m > n$.

$$A = \begin{array}{c} n \\ \left[\begin{array}{c|c|c} \vec{u}_1 & \cdots & \vec{u}_n \end{array}\right] \end{array} m$$

Then $\vec{u}_1, \ldots, \vec{u}_n$ is never a basis of $\mathbb{R}^m$.

because $\langle \vec{u}_1, \dots, \vec{u}_n \rangle \neq \mathbb{R}^m$.

Indeed, the equation $A\vec{x} = \vec{b}$ does not always have a solution, because after row reduction $A \xrightarrow{RREF} B$, B will have zero rows.

On the other hand, if $\vec{u}_1, \dots, \vec{u}_n$ are linearly independent, then they are (by Definition) a basis of the column space $\langle A \rangle \subset \mathbb{R}^m$.