Basis of a subspace

Definition: Let $W \subset \mathbb{R}^n$ be a subspace.

Let $\vec{u}_1, \dots, \vec{u}_n \in W$.

We say $\vec{u}_1, \dots, \vec{u}_n$ are a <u>basis</u> of W

if 1) $W = \langle \vec{u}_1, \dots, \vec{u}_n \rangle$

2) $\vec{u}_1, \dots, \vec{u}_n$ are linearly indep.

Example: Let $W = \mathbb{R}^2$. Let $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Claim: $\vec{u}_1, \vec{u}_2$ is a basis of W.

Proof: 1) Let $\vec{v} \in \mathbb{R}^2 = W$.

Want to show $v \in \langle \vec{u}_1, \vec{u}_2 \rangle$

$\Leftrightarrow$ there exists $\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ s.t.

$$\left[\vec{u}_1 \,\middle|\, \vec{u}_2 \right] \vec{\alpha} = \vec{v}.$$

i.e. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$

nonsingular

Can pick $\alpha_1 = v_1 \quad \alpha_2 = v_2$.

2) Want to show $\vec{u}_1, \vec{u}_2$ are linearly independent.

$\Leftrightarrow \mathcal{N}(A) = \{0_2\}$

where $A = \left[\vec{u}_1 \,\middle|\, \vec{u}_2 \right]$.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ has nullspace

$\{0_2\}$.

Hence $\vec{u}_1, \vec{u}_2$ is a basis of $W = \mathbb{R}^2$.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example 2: Let $W = \mathbb{R}^2$, $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Claim: $\vec{u}_1, \vec{u}_2$ is a basis of W.

Proof: 1) Want to show that

$$\mathbb{R}^2 = \langle \vec{u}_1, \vec{u}_2 \rangle.$$

$\iff$ for any $\vec{v} \in \mathbb{R}^2$,

there exists $\vec{\alpha} \in \mathbb{R}^2$ s.t.

$$\left[\, \vec{u}_1 \,\middle|\, \vec{u}_2 \,\right] \vec{\alpha} = \vec{v}$$

i.e. $$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \vec{\alpha} = \vec{v}.$$

Perform row reduction:

$$\left[\begin{array}{cc|c} 1 & 0 & \\ 1 & 1 & \vec{\alpha} \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \vec{\alpha}' \end{array}\right]$$

Since there are no zero rows,

equation is consistent, i.e. there exists a solution.

2) Want to show $\vec{u}_1, \vec{u}_2$ are linearly independent.

$\iff \mathcal{N}(A) = \{0_2\}$

where $A = [\vec{u}_1 | \vec{u}_2] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Can check as before that this holds. (RREF has no zero rows).

Theorem: Let $W \subset \mathbb{R}^m$ be a subspace, and let $\vec{u}_1, \dots, \vec{u}_n$ be a basis. Then given $\vec{v} \in W$, there exist unique $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ s.t.

$$\vec{v} = \alpha_1 \vec{u}_1 + \dots + \alpha_n \vec{u}_n$$

Proof: To show existence of $\alpha_1, \dots, \alpha_n$, use that $W = \langle \vec{u}_1, \dots, \vec{u}_n \rangle$.

To show uniqueness:

suppose $\vec{v} = \alpha_1 \vec{u}_1 + \dots + \alpha_n \vec{u}_n$.

$$= \beta_1 \vec{u}_1 + \dots + \beta_n \vec{u}_n$$

where $\alpha_i, \beta_i \in \mathbb{R}$.

$$\text{Then } 0_m = \vec{v} - \vec{v} = (\alpha_1 \vec{u}_1 + \dots + \alpha_n \vec{u}_n) - (\beta_1 \vec{u}_1 + \dots + \beta_n \vec{u}_n)$$

$$= (\alpha_1 - \beta_1)\vec{u}_1 + \dots + (\alpha_n - \beta_n)\vec{u}_n$$

By linear independence of $\vec{u}_1, \dots, \vec{u}_n$,
this implies $\alpha_1 - \beta_1 = 0$, $\alpha_2 - \beta_2 = 0, \dots, \alpha_n - \beta_n = 0$.

$$\iff \alpha_1 = \beta_1 \quad \alpha_2 = \beta_2, \dots, \alpha_n = \beta_n.$$

This proves uniqueness.

The theorem allows us to label vectors $\vec{v} \in W$ by $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, the scalars appearing in $\vec{v} = \alpha_1 \vec{u}_1 + \dots + \alpha_n \vec{u}_n$.

Let $A \in M_{mn}$. We can find a basis of the nullspace of A $N(A) \subset \mathbb{R}^n$ as follows:

Step 1) : If $A \xrightarrow{RREF} B$

$$\mathcal{N}(A) = \mathcal{N}(B).$$

So we will calculate

2) Let

$$B = \begin{array}{c} \begin{array}{cccccc} 1 & & 2 & & & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & * & 0 & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccccc} & 2 & & 4 & 5 & \end{array} \end{array}$$

Let r = # of pivots (= rank of B).

We will find a basis $\vec{u}_1, \dots \vec{u}_{n-r}$ of $\mathcal{N}(B)$.

To calculate $\vec{u}_i$, consider i^{th} pivotless column. look for $\vec{u}_i$ of the form

$$\vec{u}_i = \begin{bmatrix} 0 \\ * \\ 0 \\ * \\ * \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ row.}$$

nonzero only in places corresponding to pivot columns

There is a unique such vector in $\mathcal{N}(A)$.

Theorem: $\vec{u}_1, \ldots, \vec{u}_{n-r}$ form a basis of nullspace.

Early, we described $\mathcal{N}(A)$ by sets like

$$\left\{ \begin{bmatrix} a \\ a+b \\ b \\ a \end{bmatrix}, \ a, b \in \mathbb{R} \right\}.$$

Now we think of this as

$$\left\{ a\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + b\begin{bmatrix}0\\1\\1\\0\end{bmatrix},\ a,b\in\mathbb{R} \right\}.$$

$\vec{u}_1$ $\vec{u}_2$ basis

<u>Definition</u>: Let $A \in M_{mn}$.

The column space of A (sometimes called the span of A) Is the set of all vectors $\vec{x} \in \mathbb{R}^m$ of the form $A\vec{v}$ for some $\vec{v} \in \mathbb{R}^n$.

Equivalently, if $A = \left[\vec{u}_1 \middle| \cdots \middle| \vec{u}_n\right]$,

column space $= \langle \vec{u}_1, \dots, \vec{u}_n \rangle$.

Question: Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 0 & 3 \\ 2 & 1 & 3 \end{bmatrix}$.

Find a basis of the column space of A.