MATH1030 Orthogonal matrices.

1. Recall the definition for the notion of *orthogonal matrix* from the handout *Miscellanies on matrices*:

   Suppose A is an $(n \times n)$-square matrix.
   Then A is said to be orthogonal if $AA^t = I_n$ and $A^tA = I_n$.

2. Theorem (1) follows from the basic properties of matrix transpose and the basic properties of invertibility.

   Theorem (1).

   Suppose A is an $(n \times n)$-square matrix. Then the statements hold:

   (a) *If A is an orthogonal matrix, then A is invertible with matrix inverse given by A^t.*

   (b) *A is an orthogonal matrix if and only if A^t is an orthogonal matrix.*

3. Orthogonal matrices and orthonormal bases are linked up by Theorem (1).

   Theorem (2).

   Suppose A is an $(n \times n)$-square matrix, whose j-th column is denoted $\mathbf{u}_j$ for each $j = 1, 2, \cdots, n$.

   Then the statements below is are logically equivalent:

   (a) *A is an orthogonal matrix.*

   (b) *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute an orthonormal basis for $\mathbb{R}^n$.*

   Proof of Theorem (2).

   Suppose A is an $(n \times n)$-square matrix, whose j-th column is denoted $\mathbf{u}_j$ for each $j = 1, 2, \cdots, n$, and whose i-th row is denoted by $\mathbf{v}_i$ for each $i = 1, 2, \cdots, n$.

   For each $i, j = 1, 2, \cdots, n$, the (i, j)-the entry of A^tA is ${\mathbf{u}_i}^t\mathbf{u}_j$.

   - Suppose A is an orthogonal matrix.
     Then $A^tA = I_n$.
     Therefore $\langle \mathbf{u}_i, u_j \rangle = {\mathbf{u}_i}^t\mathbf{u}_j = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$.
     Hence $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute an orthonormal basis for $\mathbb{R}^n$.
   - Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute an orthonormal basis for $\mathbb{R}^n$.
     Then ${\mathbf{u}_i}^t\mathbf{u}_j = \langle \mathbf{u}_i, u_j \rangle = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$.
     Therefore $A^tA = I_n$.
     Then A is invertible with matrix inverse A^t. Hence $AA^t = I_n$ also.

4. Combining Theorem (1) and Theorem (2), we obtain Theorem (3).

   Theorem (3).

   Suppose A is an $(n \times n)$-square matrix, whose i-th row is denoted by $\mathbf{v}_i$ for each $i = 1, 2, \cdots, n$.

   Then the statements below is are logically equivalent:

   (a) *A is an orthogonal matrix.*

   (b) *${\mathbf{v}_1}^t, {\mathbf{v}_2}^t, \cdots, {\mathbf{v}_n}^t$ constitute an orthonormal basis for $\mathbb{R}^n$.*

5. **Theorem (4).**

   Let A be an $(n \times n)$-square matrix. Suppose A is an orthogonal matrix.

   Then for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\langle \mathbf{Ax}, \mathbf{Ay} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$.

   Proof of Theorem (4).

   Let A be an $(n \times n)$-square matrix. Suppose A is an orthogonal matrix.

   Pick any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

   $\langle A\mathbf{x}, A\mathbf{y} \rangle = (A\mathbf{x})^t(A\mathbf{y}) = \mathbf{y}^t(A^tA)\mathbf{x} = \mathbf{x}^tI_n\mathbf{y} = \mathbf{x}^t\mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle$.

6. **Theorem (5).**

   Suppose A is an $(n \times n)$-square matrix.

   Then the statements are logically equivalent:

   (a) *A is an orthogonal matrix.*

   (b) *For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\langle \mathbf{Ax}, \mathbf{Ay} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$.*

   (c) *For any $\mathbf{z} \in \mathbb{R}^n$, $\|A\mathbf{z}\| = \|\mathbf{z}\|$.*

   Proof of Theorem (5).

   Suppose A is an $(n \times n)$-square matrix.

   - Suppose A is an orthogonal matrix. Then for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\langle \mathbf{Ax}, \mathbf{Ay} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$.
   - Suppose that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\langle \mathbf{Ax}, \mathbf{Ay} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$.
     Pick any $\mathbf{z} \in \mathbb{R}^n$.
     Then $\|A\mathbf{z}\|^2 = \langle A\mathbf{z}, A\mathbf{z} \rangle = \langle \mathbf{z}, \mathbf{z} \rangle = \|\mathbf{z}\|^2$.
     Therefore $\|A\mathbf{z}\| = \|\mathbf{z}\|$.
   - Suppose that for any $\mathbf{z} \in \mathbb{R}^n$, $\|A\mathbf{z}\| = \|\mathbf{z}\|$.
     We deduce that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\langle \mathbf{Ax}, \mathbf{Ay} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$.
     In particular for each $i, j = 1, 2, \cdots, n$, $\left\langle \mathbf{Ae_i^{(n)}}, \mathbf{Ae_j^{(n)}} \right\rangle = \left\langle \mathbf{e}_i^{(n)}, \mathbf{e}_j^{(n)} \right\rangle = \left\{ \begin{array}{lll} 1 & \text{if} & i = j \\ 0 & \text{if} & i \neq j \end{array} \right.$.
     Then the n columns of A, namely, $A\mathbf{e}_1^{(n)}, A\mathbf{e}_2^{(n)}, \cdots, \mathbf{e}_n^{(n)}$ constitute an orthonormal basis for $\mathbb{R}^n$.
     It follows that A is an orthogonal matrix.

7. **Theorem (6).**

   Let A, B be $(n \times n)$-square matrices. Suppose A, B are orthogonal matrices. Then AB is an orthogonal matrix.

   Proof of Theorem (6).

   Let A, B be $(n \times n)$-square matrices. Suppose A, B are orthogonal matrices.

   Then $A^t A = I_n = AA^t$ and $B^t B = I_n = BB^t$.

   We have $(AB)^t(AB) = (B^t A^t)(AB) = B^t(AA^t)B = B^t I_n B = B^t B = I_n$.

   Also $(AB)(AB)^t = (AB)(B^t A^t) = A(BB^t)A^t = AI_n A^t = AA^t = I_n$.

   Therefore AB is an orthogonal matrix.

8. **Theorem (7).**

   Let A be an $(n \times n)$-square matrix, and $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ be vectors in $\mathbb{R}^n$.

   Suppose A is an orthogonal matrix, and $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ constitute an orthonormal basis for $\mathbb{R}^n$.

   Then $A\mathbf{v}_1, A\mathbf{v}_2, \cdots, A\mathbf{v}_n$ constitute a basis for $\mathbb{R}^n$.

   Proof of Theorem (7).

   Let A be an $(n \times n)$-square matrix, and $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ be vectors in $\mathbb{R}^n$.

   Suppose A is an orthogonal matrix, and $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ constitute an orthonormal basis for $\mathbb{R}^n$.

   Define $B = [\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_n\ ]$. By assumption, B is an orthogonal matrix.

   Then by Theorem (6), AB is an orthgonal matrix. Its columns are $A\mathbf{v}_1, A\mathbf{v}_2, \cdots, A\mathbf{v}_n$. By Theorem (2), they constitute an orthonormal basis for $\mathbb{R}^n$.

9. **Theorem (8).**

   Let A be an $(n \times n)$-square matrix. Suppose A is an orthogonal matrix. Then $\det(A) = 1$ or $\det(A) = -1$.

   Proof of Theorem (8).

   Let A be an $(n \times n)$-square matrix. Suppose A is an orthogonal matrix.

   Then $A^t A = I_n$

   Therefore $1 = \det(A^t A) = \det(A^t)\det(A) = \det(A) \cdot \det(A) = (\det(A))^2$.

   Hence $\det(A) = 1$ or $\det(A) = -1$.