Example (A)

Recall that an $(n \times n)$-square matrix A is said to be orthogonal if

$$A^t A = I_n = A A^t.$$

① Let A be an orthogonal $(n \times n)$-square matrix.

Show that $\det(A) = 1$ or $\det(A) = -1$.

Remark.

If $\det(A) = 1$ then such a orthogonal matrix is called a special orthogonal $(n \times n)$-square matrix.

② Let A, B be orthogonal $(n \times n)$-square matrices.

(2a) Show that

$$\det(A^2 + AB) = \det(I_n + A^t B) = \det(B^2 + AB).$$

(2b) (This is independent of (2a).)

Suppose $\det(A)\det(B) = -1$.

Show that

$$\begin{cases} \det(A^2 + AB) = \det(I_n + AB^t) \\ \det(A^2 + AB) = -\det(I_n + AB^t) \end{cases}.$$

Hence deduce that $A + B$ is singular.

Example (B)

Recall that an $(n \times n)$-square matrix A is said to be skew-symmetric if $A^t = -A$.

① Let A be an skew-symmetric $(n \times n)$-square matrix.

Suppose n is odd.

Show that A is singular.

② Note that the diagonal entries of a skew-symmetric matrix are all zero. (How comes?)

(2a) Evaluate $\det\left(\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}\right)$ for each $a \in \mathbb{R}$.

(2b) Show that for each $b_1, b_2, b_3, e_1, e_2, e_3 \in \mathbb{R}$,

$$\det\left(\begin{bmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & b_3 & -b_2 \\ -e_2 & -b_3 & 0 & b_1 \\ -e_3 & b_2 & -b_1 & 0 \end{bmatrix}\right) = (e_1 b_1 + e_2 b_2 + e_3 b_3)^2$$

Example (C)

① Suppose A is an $(n\times n)$-square matrix, and B is a $(p\times p)$-square matrix.

Suppose C is the $(n+p)\times(n+p)$-square matrix given by

$$C = \left[\begin{array}{c|c} A & O_{n\times p} \\ \hline O_{p\times n} & B \end{array}\right].$$

ⓐ Suppose B is singular.
Show that $\det(C) = 0$.

ⓑ Suppose $B = I_p$.
Show that $\det(C) = \det(A)$.

ⓒ Suppose $A = I_n$.
Show that $\det(C) = \det(B)$.

ⓓ Show that $\det(C) = \det(A)\det(B)$.
(Hint. Apply ⓐ, ⓑ, ⓒ.)

② Let A, B be $(n\times n)$-square matrices.
Show that

$$\det\left(\left[\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right]\right) = \det(A+B)\det(A-B)$$

③ Let A be an $(n\times n)$-square matrix, D be $(p\times p)$-square matrix, and B be a $(n\times p)$-matrix.
Show that

$$\det\left(\left[\begin{array}{c|c} A & B \\ \hline O & D \end{array}\right]\right) = \det(A)\det(D)$$

Hint: Split the argument into two cases:

ⓐ A or D is singular.

ⓑ Neither of A, D is singular.