Further Example (1A)

Evaluate $\det\left(\begin{bmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{bmatrix}\right)$ for each $a, b, c \in \mathbb{R}$.

$$\det\left(\begin{bmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1 & a & b+c \\ 0 & b-a & a-b \\ 0 & c-a & a-c \end{bmatrix}\right)$$

$$= (b-a)(c-a)\det\left(\begin{bmatrix} 1 & a & b+c \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}\right)$$

$$= 0$$

Further Example (1B)

Evaluate $\det\left(\begin{bmatrix} a & b & a+b \\ b & a+b & a \\ a+b & a & b \end{bmatrix}\right)$ for each $a, b \in \mathbb{R}$.

$$\det\left(\begin{bmatrix} a & b & a+b \\ b & a+b & a \\ a+b & a & b \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} a & b & a+b \\ b & a+b & a \\ 0 & -2b & -2a \end{bmatrix}\right)$$

$$= -2\det\left(\begin{bmatrix} a & b & a+b \\ b & a+b & a \\ 0 & b & a \end{bmatrix}\right)$$

$$= -2\det\left(\begin{bmatrix} a & 0 & b \\ b & a & 0 \\ 0 & b & a \end{bmatrix}\right)$$

$$= -2(a^3+b^3)$$

$$= -2(a+b)(a^2-ab+b^2)$$

Alternative method.

$$\det\left(\begin{bmatrix} a & b & a+b \\ b & a+b & a \\ a+b & a & b \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 2(a+b) & 2(a+b) & 2(a+b) \\ b & a+b & a \\ a+b & a & b \end{bmatrix}\right)$$

$$= 2(a+b)\det\left(\begin{bmatrix} 1 & 1 & 1 \\ b & a+b & a \\ a+b & a & b \end{bmatrix}\right)$$

$$= 2(a+b)\det\left(\begin{bmatrix} 1 & 0 & 0 \\ b & a & a-b \\ a+b & -b & -a \end{bmatrix}\right)$$

$$= 2(a+b)\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & a-b \\ 2b & -b & -a \end{bmatrix}\right)$$

$$= 2(a+b)\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & a-b \\ 0 & -b & -a \end{bmatrix}\right)$$

$$= 2(a+b)\cdot\left[1\cdot a\cdot(-a) - 1\cdot(a-b)\cdot(-b)\right]$$

$$= -2(a+b)(a^2-ab+b^2)$$

Further Example (1C)

Evaluate $\det\left(\begin{bmatrix} a^2 & 1-a^3 & 2a^3 \\ b^2 & 1-b^3 & 2b^3 \\ c^2 & 1-c^3 & 2c^3 \end{bmatrix}\right)$ for each $a,b,c\in\mathbb{R}$

$$\det\left(\begin{bmatrix} a^2 & 1-a^3 & 2a^3 \\ b^2 & 1-b^3 & 2b^3 \\ c^2 & 1-c^3 & 2c^3 \end{bmatrix}\right)$$

$$= 2\det\left(\begin{bmatrix} a^2 & 1-a^3 & a^3 \\ b^2 & 1-b^3 & b^3 \\ c^2 & 1-c^3 & c^3 \end{bmatrix}\right)$$

$$= 2\det\left(\begin{bmatrix} a^2 & 1 & a^3 \\ b^2 & 1 & b^3 \\ c^2 & 1 & c^3 \end{bmatrix}\right)$$

$$= 2\det\left(\begin{bmatrix} a^2 & 1 & a^3 \\ b^2-a^2 & 0 & b^3-a^3 \\ c^2-a^2 & 0 & c^3-a^3 \end{bmatrix}\right)$$

$$= 2(b-a)(c-a)\det\left(\begin{bmatrix} a^2 & 1 & a^3 \\ b+a & 0 & b^2+ab+a^2 \\ c+a & 0 & c^2+ac+a^2 \end{bmatrix}\right)$$

$$= 2(b-a)(c-a)\det\left(\begin{bmatrix} a^2 & 1 & a^3 \\ b-c & 0 & b^2-c^2+ab-ac \\ c+a & 0 & c^2+ac+a^2 \end{bmatrix}\right)$$

$$= 2(b-a)(c-a)(b-c)\det\left(\begin{bmatrix} a^2 & 1 & a^3 \\ 1 & 0 & b+c+a \\ c+a & 0 & c^2+ac+a^2 \end{bmatrix}\right)$$

$$= 2(b-a)(c-a)(b-c)\det\left(\begin{bmatrix} a^2 & 1 & 0 \\ 1 & 0 & b+c \\ c+a & 0 & c^2 \end{bmatrix}\right)$$

$$= 2(b-a)(c-a)(b-c)\det\left(\begin{bmatrix} a^2 & 1 & 0 \\ 1 & 0 & b+c \\ a & 0 & -bc \end{bmatrix}\right)$$

$$= 2(b-a)(c-a)(b-c)\det\left(\begin{bmatrix} a^2 & 1 & 0 \\ 1 & 0 & b+c \\ 0 & 0 & -ab-bc-ca \end{bmatrix}\right)$$

$$= -2(b-a)(c-a)(b-c)\det\left(\begin{bmatrix} 1 & a^2 & 0 \\ 0 & 1 & b+c \\ 0 & 0 & -ab-bc-ca \end{bmatrix}\right) = -2(a-b)(b-c)(c-a)(ab+bc+ca)$$

Further Example (1D)

Evaluate $\det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix}\right)$ for each $a,b,c,d\in\mathbb{R}$.

$$\det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & b-a & c-a & d-a \\ a^2 & b^2-a^2 & c^2-a^2 & d^2-a^2 \\ a^3 & b^3-a^3 & c^3-a^3 & d^3-a^3 \end{bmatrix}\right)$$

$$= (b-a)(c-a)(d-a)\cdot\det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 1 & 1 \\ a^2 & b+a & c+a & d+a \\ a^3 & b^2+ab+a^2 & c^2+ac+a^2 & d^2+ad+a^2 \end{bmatrix}\right)$$

$$= (b-a)(c-a)(d-a)\cdot\det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & b+a & c-b & d-b \\ a^3 & b^2+ab+a^2 & c^2-b^2+ac-ab & d^2-b^2+ad-ab \end{bmatrix}\right)$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)\cdot\det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & b+a & 1 & 1 \\ a^3 & b^2+ab+a^2 & a+b+c & a+b+d \end{bmatrix}\right)$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)\det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & b+a & 1 & 0 \\ a^3 & b^2+ab+a^2 & a+b+c & d-c \end{bmatrix}\right)$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)(d-c).$$

Further Example (2)

① Show that

$$\det\left(\begin{bmatrix} a^2 & ab & b^2 \\ b^2 & bc & c^2 \\ c^2 & ca & a^2 \end{bmatrix}\right) = (a^2-bc)(b^2-ca)(c^2-ab)$$

for each $a, b, c \in \mathbb{R}$.

② Show that

$$\det\left(\begin{bmatrix} a^2 & bc & c^2+ca \\ a^2+ab & b^2 & ca \\ ab & b^2+bc & c^2 \end{bmatrix}\right) = 4a^2b^2c^2$$

for each $a, b, c \in \mathbb{R}$.

③ Show that

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix}\right) = (b-c)(c-a)(a-b)(a+b+c)$$

for each $a, b, c \in \mathbb{R}$. Hence, or otherwise, evaluate

$$\det\left(\begin{bmatrix} a & b & c \\ b+c & c+a & a+b \\ a^3 & b^3 & c^3 \end{bmatrix}\right)$$ for each $a, b, c \in \mathbb{R}$.

④ Show that

$$\det\left(\begin{bmatrix} c & a & b \\ c^2 & a^2 & b^2 \\ a+b & b+c & c+a \end{bmatrix}\right) = (b-a)(c-a)(c-b)(a+b+c)$$

for each $a, b, c \in \mathbb{R}$.

⑤ Show that

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ bc(c-b) & ca(a-c) & ab(b-a) \\ b^2c & c^2a & a^2b \end{bmatrix}\right) = abc(a^3+b^3+c^3-3ab)$$

for each $a, b, c \in \mathbb{R}$.

⑥ Suppose $a^2+b^2+c^2+d^2=1$. Show that

$$\det\left(\begin{bmatrix} a^2-1 & ab & ac & ad \\ ba & b^2-1 & bc & bd \\ ca & cb & c^2-1 & cd \\ da & db & dc & d^2-1 \end{bmatrix}\right) = 0.$$

⑦ Suppose a, b, c, d are all non-zero. Show that

$$\det\left(\begin{bmatrix} a+1 & 1 & 1 & 1 \\ 1 & b+1 & 1 & 1 \\ 1 & 1 & c+1 & 1 \\ 1 & 1 & 1 & d+1 \end{bmatrix}\right) = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

Remark. How about $\det\left(\begin{bmatrix} a+1 & 1 & \cdots & & 1 \\ 1 & b+1 & & \text{all 1's} & \vdots \\ \vdots & \text{all 1's} & c+1 & & \\ & & & d+1 & 1 \\ 1 & \cdots & & 1 & e+1 \end{bmatrix}\right)$?

(8) Suppose n is an integer greater than 1. Show that

$$\det\left(\begin{bmatrix} (n-2)! & (n-1)! & n! \\ (n-1)! & n! & (n+1)! \\ n! & (n+1)! & (n+2)! \end{bmatrix}\right) = 2\left[(n-2)!\right]\left[(n-1)!\right](n!)$$

(9) Show that

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ \cos(2\alpha) & \cos(2\beta) & \cos(2\gamma) \\ \sin(2\alpha) & \sin(2\beta) & \sin(2\gamma) \end{bmatrix}\right) = 4\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$.

(10) Show that

$$\det\left(\begin{bmatrix} 1 & \cos(2\alpha) & \sin(\alpha) \\ 1 & \cos(2\beta) & \sin(\beta) \\ 1 & \cos(2\gamma) & \sin(\gamma) \end{bmatrix}\right) = 2(\sin(\beta)-\sin(\gamma))(\sin(\gamma)-\sin(\alpha))(\sin(\alpha)-\sin(\beta))$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$.

(11) Show that

$$\det\left(\begin{bmatrix} \sin(2\alpha) & \sin(\alpha+\beta) & \sin(\beta) \\ \sin(\alpha+\beta) & \sin(2\beta) & \sin(\alpha) \\ \sin(\beta) & \sin(\alpha) & -\sin(2\alpha+2\beta) \end{bmatrix}\right) = 0$$

for any $\alpha, \beta \in \mathbb{R}$.

(12) Show that

$$\det\left(\begin{bmatrix} 1 & u & u^2 \\ \cos((n-1)\alpha) & \cos(n\alpha) & \cos((n+1)\alpha) \\ \sin((n-1)\alpha) & \sin(n\alpha) & \sin((n+1)\alpha) \end{bmatrix}\right) = (1-2u\cos(\alpha)+u^2)\sin(\alpha)$$

for any $\alpha, u \in \mathbb{R}$, for any integer n.