MATH1030 Further examples on characteristic polynomials, eigenvalues and eigenspaces.

1. For each matrix below, determine its characteristic polynomial and its eigenvalues. Furthermore, for each eigenvalue, determine a basis for the corresponding eigenspace. Also determine whether the matrix concerned is diagaonalizable.

(a) $\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & -4 \\ 4 & -3 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & 8 \\ 1 & 7 \end{bmatrix}$

(d) $\begin{bmatrix} -8 & 7 \\ -4 & 8 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 2 \\ -2 & -3 \end{bmatrix}$

(f) $\begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -1 & -1 \end{bmatrix}$

(i) $\begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

(j) $\begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

(k) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ -3 & -3 & 5 \end{bmatrix}$

(l) $\begin{bmatrix} -3 & -2 & 2 \\ 4 & 3 & -4 \\ -2 & -2 & 1 \end{bmatrix}$

2. (a) Let $\{x_n\}_{n=0}^{\infty}$ be the infinite sequence of real numbers defined recursively by

$$\begin{cases} x_0 & = & 0 \\ x_1 & = & 1 \\ x_{n+2} & = & 2x_{n+1} + 8x_n \quad \text{for any natural number } n \end{cases}$$

i. Write down the $(2 \times 2)$-square matrix $A = \begin{bmatrix} \alpha & \beta \\ 1 & 0 \end{bmatrix}$ for which the equality $\begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = A \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$ holds for every natural number $n$.
Here $\alpha, \beta$ are some appropriate real numbers, independent of $n$, which you have to determine explicitly.

ii. Find the characteristic polynomial $p_A(x)$, and find the eigenvalues of $A$.

iii. Hence find a diagaonalization for $A$, and show that $A^n = \frac{\lambda^n}{6}\begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix} + \frac{\mu^n}{6}\begin{bmatrix} 2 & -8 \\ -1 & 4 \end{bmatrix}$ for every natural number $n$.
Here $\lambda, \mu$ are some appropriate real numbers, independent of $n$, which you have to determine explicitly.

iv. Hence find an explicit formula for $x_n$ (in terms of $n$ alone).

(b) Imitate the process described above to find an explicit formula for the individual terms of each recursively defined infinite sequence described below:

i. $\begin{cases} x_0 & = & 0 \\ x_1 & = & 6 \\ x_{n+2} & = & x_{n+1} + 2x_n \quad \text{for any natural number } n \end{cases}$

ii. $\begin{cases} x_0 & = & -1 \\ x_1 & = & 1 \\ x_{n+2} & = & 5x_{n+1} - 6x_n \quad \text{for any natural number } n \end{cases}$

iii. $\begin{cases} x_0 & = & 3 \\ x_1 & = & 6 \\ x_2 & = & 14 \\ x_{n+3} & = & 6x_{n+2} - 11x_{n+1} + 6x_n \quad \text{for any natural number } n \end{cases}$

Answer.

1. (a) Characteristic polynomial: $x(x-4)$.
Eigenvalues: $0, 4$.
A basis for eigenspace with eigenvalue 0 is given by: $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.
A basis for eigenspace with eigenvalue 4 is given by: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
The matrix is diagonalizable.

(b) Characteristic polynomial: $(x-1)^2$.
Eigenvalue: 1 only.
A basis for eigenspace with eigenvalue 4 is given by: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
The matrix is not diagonalizable.

(c) Characteristic polynomial: $(x-3)(x-9)$.
Eigenvalues: $3, 9$.
A basis for eigenspace with eigenvalue 3 is given by: $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$.
A basis for eigenspace with eigenvalue 9 is given by: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
The matrix is diagonalizable.

(d) Characteristic polynomial: $(x+6)(x-6)$.
Eigenvalues: $-6, 6$.
A basis for eigenspace with eigenvalue $-6$ is given by: $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$.
A basis for eigenspace with eigenvalue 6 is given by: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
The matrix is diagonalizable.

(e) Characteristic polynomial: $(x+\sqrt{5})(x-\sqrt{5})$.
Eigenvalues: $-\sqrt{5}, \sqrt{5}$.
A basis for eigenspace with eigenvalue $-\sqrt{5}$ is given by: $\begin{bmatrix} (-3+\sqrt{5})/2 \\ 1 \end{bmatrix}$.
A basis for eigenspace with eigenvalue $\sqrt{5}$ is given by: $\begin{bmatrix} (-3-\sqrt{5})/2 \\ 1 \end{bmatrix}$.
The matrix is diagonalizable.

(f) Characteristic polynomial: $x^2 - 4x + 8$.
The matrix has no eigenvalues, and is not diagonalizable.

(g) Characteristic polynomial: $-(x+1)(x-1)(x-3)$.
Eigenvalues: $-1, 1, 3$.
A basis for eigenspace with eigenvalue $-1$ is given by: $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.
A basis for eigenspace with eigenvalue 1 is given by: $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.
A basis for eigenspace with eigenvalue 3 is given by: $\begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$.
The matrix is diagonalizable.

(h) Characteristic polynomial: $-(x+1)(x-1)(x-2)$.
Eigenvalues: $-1, 1, 2$.
A basis for eigenspace with eigenvalue $-1$ is given by: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
A basis for eigenspace with eigenvalue 1 is given by: $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.
A basis for eigenspace with eigenvalue 2 is given by: $\begin{bmatrix} -6/5 \\ -3/5 \\ 1 \end{bmatrix}$.
The matrix is diagonalizable.

(i) Characteristic polynomial:$-(x-1)(x-2)(x-3)$.
Eigenvalues: $1, 2, 3$.

A basis for eigenspace with eigenvalue 1 is given by: $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

A basis for eigenspace with eigenvalue 2 is given by: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

A basis for eigenspace with eigenvalue 3 is given by: $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

The matrix is diagonalizable.

(j) Characteristic polynomial: $-(x-1)^2(x-2)$.
Eigenvalues: $1, 2$.

A basis for eigenspace with eigenvalue 1 is given by: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$.

A basis for eigenspace with eigenvalue 2 is given by: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

The matrix is diagonalizable.

(k) Characteristic polynomial: $-(x-2)^2(x-6)$.
Eigenvalues: $2, 6$.

A basis for eigenspace with eigenvalue 2 is given by: $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

A basis for eigenspace with eigenvalue 6 is given by: $\begin{bmatrix} 1/3 \\ -2/3 \\ 1 \end{bmatrix}$.

The matrix is diagonalizable.

(l) Characteristic polynomial: $-(x+1)^2(x-3)$.
Eigenvalues: $-1, 3$.

A basis for eigenspace with eigenvalue $-1$ is given by: $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

A basis for eigenspace with eigenvalue 3 is given by: $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

The matrix is diagonalizable.

2. (a) i. $\alpha = 2, \beta = 8$.

ii. $p_A(x) = x^2 - 2x - 8$.
The eigenvalues of $A$ are $-2, 4$.

iii. A diagonalization of $A$ is given by $U^{-1}AU = \text{diag}(4, -2)$, in which $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2\ ]$, $\mathbf{u}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
$\lambda = 4$, $\mu = -2$.

iv. $x_n = \dfrac{4^n}{6} + (-1)^{n-1} \cdot \dfrac{2^{n-1}}{3}$ for each natural number $n$.

(b) i. $x_n = 2(-1)^{n+1} + 2^{n+1}$ for each natural number $n$.

ii. $x_n = -2^{n+2} + 3^{n+1}$ for each natural number $n$.

iii. $x_n = 1 + 2^n + 3^n$ for each natural number $n$.