

1. **Lemma (1).**

Let H, B be $(n \times n)$ -square matrices. Suppose H is a row-operation matrix. Then $\det(HB) = \det(H) \det(B)$.

Proof of Lemma (1).

Let H, B be $(n \times n)$ -square matrices. Suppose H is a row-operation matrix. There are three possibilities:

- H is the row operation matrix corresponding to the row operation $\alpha R_i + R_k$ for some distinct i, k and for some real number α .
- H is the row operation matrix corresponding to the row operation βR_i for some non-zero real number β .
- H is the row operation matrix corresponding to the row operation $R_i \leftrightarrow R_k$ for some distinct i, k .

(a) Suppose H is the row operation matrix corresponding to the row operation $\alpha R_i + R_k$ for some distinct i, k and for some real number α .

Then $\det(H) = 1$.

HB is obtained by B by adding a scalar multiple of the i -th row to the k -th row. Then $\det(HB) = \det(B)$.

Therefore $\det(HB) = 1 \cdot \det(B) = \det(H) \det(B)$.

(b) Suppose H is the row operation matrix corresponding to the row operation βR_i for some non-zero real number β .

Then $\det(H) = \beta$.

HB is obtained by B by multiplying every entry of the i -th row by β . Then $\det(HB) = \beta \det(B)$.

Therefore $\det(HB) = \beta \det(B) = \det(H) \det(B)$.

(c) Suppose H is the row operation matrix corresponding to the row operation $R_i \leftrightarrow R_k$ for some distinct i, k .

Then $\det(H) = -1$.

HB is obtained by B by interchanging the i -th row and the k -th row. Then $\det(HB) = -\det(B)$.

Therefore $\det(HB) = -\det(B) = \det(H) \det(B)$.

Hence, in any case, $\det(HB) = \det(H) \det(B)$.

2. **Corollary to Lemma (1).**

Let H_1, H_2, \dots, H_k be $(n \times n)$ -matrices. Suppose H_1, H_2, \dots, H_k are row operation matrices.

Then

$$\det(H_k H_{k-1} \cdots H_2 H_1) = \det(H_k) \det(H_{k-1}) \cdots \det(H_2) \det(H_1).$$

Proof of Corollary to Lemma (1).

Let H_1, H_2, \dots, H_k be $(n \times n)$ -matrices.

Suppose H_1, H_2, \dots, H_k are row operation matrices.

Then

$$\begin{aligned} \det(H_k H_{k-1} \cdots H_2 H_1) &= \det(H_k) \det(H_{k-1} \cdots H_2 H_1) \\ &= \det(H_k) \det(H_{k-1}) \det(H_{k-2} \cdots H_2 H_1) \\ &= \cdots \\ &= \det(H_k) \det(H_{k-1}) \cdots \det(H_3) \det(H_2 H_1) \\ &= \det(H_k) \det(H_{k-1}) \cdots \det(H_2) \det(H_1). \end{aligned}$$

3. **Theorem (2).**

Let A, B be $(n \times n)$ -square matrices. Suppose A is nonsingular. Then $\det(AB) = \det(A) \det(B)$.

Proof of Theorem (2).

Let A, B be $(n \times n)$ -square matrices. Suppose A is nonsingular.

Then there are some k row-operation matrices, say, H_1, H_2, \dots, H_k , so that $A = H_k H_{k-1} \cdots H_2 H_1$.

Therefore

$$\begin{aligned}\det(AB) &= \det(H_k H_{k-1} \cdots H_2 H_1 B) \\ &= \det(H_k) \det(H_{k-1} \cdots H_2 H_1 B) \\ &= \det(H_k) \det(H_{k-1}) \det(H_{k-2} \cdots H_2 H_1 B) \\ &= \cdots \\ &= \det(H_k) \det(H_{k-1}) \cdots \det(H_2) \det(H_1 B) \\ &= \det(H_k) \det(H_{k-1}) \cdots \det(H_2) \det(H_1) \det(B) \\ &= \det(H_k H_{k-1} \cdots H_2 H_1) \det(B) = \det(A) \det(B)\end{aligned}$$

Then $\det(AB) = \det(A) \det(B)$.

4. **Lemma (3).**

Let C be an $(n \times n)$ -square matrix. Suppose C is singular.

Then $\det(C) = 0$.

Proof of Lemma (3).

Let C be an $(n \times n)$ -square matrix. Suppose C is singular.

Denote by C' the reduced row-echelon form which is row-equivalent to C .

Since C is singular, C' is also singular. (Why?) Then, since C' is a singular reduced row-echelon form, C' has at least one entire row of 0's. Therefore $\det(C') = 0$.

Since C is row-equivalent to C' , there is some non-singular $(n \times n)$ -square matrix A such that $C = AC'$.

Then, by Theorem (2),

$$\det(C) = \det(AC') = \det(A) \det(C') = 0.$$

5. **Theorem (4).**

Let A, B be $(n \times n)$ -square matrices. Suppose A is singular. Then $\det(AB) = 0 = \det(A) \det(B)$.

Proof of Theorem (4).

Let A, B be $(n \times n)$ -square matrices. Suppose A is singular.

Then by Theorem (3), we have $\det(A) = 0$. Therefore $\det(A) \det(B) = 0$.

Since A is singular, AB is also singular. Then by Theorem (3), we have $\det(AB) = 0$.

Therefore $\det(AB) = 0 = \det(A) \det(B)$.

6. Combining Theorem (2) and Theorem (4), we obtain the result below:

Theorem (ζ).

Suppose A, B are $(n \times n)$ -square matrices. Then $\det(AB) = \det(A) \det(B)$.

Remark. Actually it further follows that $\det(AB) = \det(A) \det(B) = \det(B) \det(A) = \det(BA)$. However, note that AB and BA are not necessarily the same matrix.

7. An immediate consequence of Theorem (ζ) is Theorem (η).

Theorem (η).

Suppose A is an $(n \times n)$ -square matrix. Then the statements below holds:

- (a) For any positive integer p , $\det(A^p) = (\det(A))^p$.
- (b) Suppose A is invertible. Then $\det(A) \neq 0$, and $\det(A^{-1}) = (\det(A))^{-1}$.

8. Statement (b) in Theorem (η) tells us that if a square matrix is invertible then its determinant is non-zero.

It is natural to ask whether it is true that if the determinant of a square matrix is non-zero then the matrix concerned is invertible. The answer is provided by Theorem (5).

Theorem (5).

Let A be an $(n \times n)$ -square matrix. Suppose $\det(A) \neq 0$. Then A is invertible.

Proof of Theorem (5).

Let A be an $(n \times n)$ -square matrix. Suppose $\det(A) \neq 0$.

[We want to deduce that A is non-singular. How? We try to show that A is row-equivalent to I_n .]

Denote by A' the reduced row-echelon form which is row-equivalent to A .

[Ask: Is it true that $A' = I_n$? To find the answer, we ask whether $\det(A') \neq 0$ or not.]

There exists some non-singular $(n \times n)$ -square matrix H such that $A' = HA$.

By Theorem (ζ), we have $\det(A') = \det(H) \det(A)$.

Since H is non-singular, we have $\det(H) \neq 0$. By assumption, $\det(A) \neq 0$. Then $\det(A') \neq 0$.

By assumption A' is a reduced row-echelon form. Since $\det(A') \neq 0$, there is no row of A' which is a row of 0's. Then every row of A' contains a leading one. Therefore $A' = I_n$.

Hence A is row equivalent to I_n . Then A is non-singular.

9. Combining Theorem (η) and Theorem (5), we obtain the result below:

Theorem (θ).

Suppose A is an $(n \times n)$ -square matrix. Then the statements below are logically equivalent:

- (a) A is non-singular.
- (b) A is invertible.
- (c) $\det(A) \neq 0$.

10. **Corollary to Theorem (θ).**

Suppose A is an $(n \times n)$ -square matrix. Then the statements below are logically equivalent:

- (a) A is singular.
- (b) A is not invertible.
- (c) $\det(A) = 0$.

11. We now compile and re-organized all the various re-formulations for the notions of non-singularity and invertibility that we have learnt so far into one single result:

Theorem (ι). (Various re-formulations for the notions of non-singularity and invertibility.)

Let A be an $(n \times n)$ -matrix.

(a) The statements below are logically equivalent:

- i. A is non-singular.
- ii. For any vector \mathbf{v} in \mathbb{R}^n , if $A\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$.
- iii. The trivial solution is the only solution of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.
- iv. A is row-equivalent to I_n .
- v. A is invertible.
- vi. There exists some $(n \times n)$ -square matrix H such that $HA = I_n$.
- vii. There exists some $(n \times n)$ -square matrix G such that $AG = I_n$.
- viii. For any vector \mathbf{b} in \mathbb{R}^n , the system $\mathcal{LS}(A, \mathbf{b})$ has one and only one solution, namely, ' $\mathbf{x} = A^{-1}\mathbf{b}$ '.
- ix. For any vector \mathbf{c} in \mathbb{R}^n , the system $\mathcal{LS}(A, \mathbf{c})$ has at least one solution.
- x. For any vector \mathbf{d} in \mathbb{R}^n , the system $\mathcal{LS}(A, \mathbf{d})$ has at most one solution.

(b) The statements below are logically equivalent:

- i. A is non-singular.
- ii. A^t is non-singular.
- iii. For any vector \mathbf{v} in \mathbb{R}^n , if $A^t\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$.
- iv. The trivial solution is the only solution of the homogeneous system $\mathcal{LS}(A^t, \mathbf{0})$.
- v. A^t is row-equivalent to I_n .
- vi. A^t is invertible.
- vii. There exists some $(n \times n)$ -square matrix J such that $JA^t = I_n$.
- viii. There exists some $(n \times n)$ -square matrix K such that $A^tK = I_n$.
- ix. For any vector \mathbf{b} in \mathbb{R}^n , the system $\mathcal{LS}(A^t, \mathbf{b})$ has one and only one solution, namely, ' $\mathbf{x} = (A^t)^{-1}\mathbf{b}$ '.
- x. For any vector \mathbf{c} in \mathbb{R}^n , the system $\mathcal{LS}(A^t, \mathbf{c})$ has at least one solution.
- xi. For any vector \mathbf{d} in \mathbb{R}^n , the system $\mathcal{LS}(A^t, \mathbf{d})$ has at most one solution.

(c) Denote the j -th column of A by \mathbf{u}_j for each $j = 1, 2, \dots, n$.

The statements below are logically equivalent:

- i. A is non-singular.
- ii. Every vector in \mathbb{R}^n is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.
- iii. $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are linearly independent.
- iv. $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ constitute a basis for \mathbb{R}^n .
- v. The dimension of the column space of A is n .
- vi. The dimension of the null space of A is 0.
- vii. $\det(A) \neq 0$.

(d) Denote the i -th row of A by \mathbf{w}_i for each $i = 1, 2, \dots, n$.

The statements below are logically equivalent:

- i. A is non-singular.
- ii. A^t is non-singular.
- iii. Every vector in \mathbb{R}^n is a linear combination of $\mathbf{w}_1^t, \mathbf{w}_2^t, \dots, \mathbf{w}_n^t$.
- iv. $\mathbf{w}_1^t, \mathbf{w}_2^t, \dots, \mathbf{w}_n^t$ are linearly independent.
- v. $\mathbf{w}_1^t, \mathbf{w}_2^t, \dots, \mathbf{w}_n^t$ constitute a basis for \mathbb{R}^n .
- vi. The dimension of the row space of A is n .
- vii. The dimension of the null space of A^t is 0.
- viii. $\det(A^t) \neq 0$.

(e) Now further suppose A is non-singular, with a sequence of row operations

$$A = C_1 \xrightarrow{\rho_1} C_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{p-2}} C_{p-1} \xrightarrow{\rho_{p-1}} C_p = I_n,$$

and with H_k being the row-operation matrix corresponding to ρ_k for each k . Then $[I_n|A^{-1}]$ is the resultant of the application of the same sequence of row operations $\rho_1, \rho_2, \dots, \rho_{p-1}$ starting from $[A|I_n]$:

$$[A|I_n] = [C_1|I_n] \xrightarrow{\rho_1} [C_2|H_1] \xrightarrow{\rho_2} [C_3|H_2H_1] \xrightarrow{\rho_3} \cdots \xrightarrow{\rho_{p-2}} [C_{p-1}|H_{p-2} \cdots H_2H_1] \xrightarrow{\rho_{p-1}} [C_p|H_{p-1} \cdots H_2H_1] = [I_n|A^{-1}].$$

Moreover, A^{-1} and A are respectively given as products of row-operation matrices by

$$A^{-1} = H_{p-1} \cdots H_2H_1, \quad A = H_1^{-1}H_2^{-1} \cdots H_{p-1}^{-1}.$$