MATH1030 Multilinearity and alternating property of determinants.

1. **Theorem (β). (Multilinearity of determinants in columns.)**

   Let A, B, C be $(n \times n)$-square matrix, whose j-th columns are denoted by $\mathbf{a}_j, \mathbf{b}_j, \mathbf{c}_j$ respectively for each j.

   Suppose β, γ are real numbers, and there is some $q = 1, 2, \cdots, n$ so that:

   (a) $\mathbf{a}_q = \beta\mathbf{b}_q + \gamma\mathbf{c}_q$, *and*

   (b) $\mathbf{a}_j = \mathbf{b}_j = \mathbf{c}_j$ *whenever* $j \neq q$.

   Then $\det(A) = \beta\det(B) + \gamma\det(C)$.

   Remark. Presented in symbols, what happens is:

$$\begin{aligned} & \det([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{a}_{q-1} \mid \beta\mathbf{b}_q + \gamma\mathbf{c}_q \mid \mathbf{a}_{q+1} \mid \cdots \mid \mathbf{a}_n\ ]) \\ = \ & \beta \cdot \det([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{a}_{q-1} \mid \mathbf{b}_q \mid \mathbf{a}_{q+1} \mid \cdots \mid \mathbf{a}_n\ ]) + \gamma \cdot \det([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{a}_{q-1} \mid \mathbf{c}_q \mid \mathbf{a}_{q+1} \mid \cdots \mid \mathbf{a}_n\ ]) \end{aligned}$$

   In particular,

$$\det([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{a}_{q-1} \mid \beta\mathbf{b}_q \mid \mathbf{a}_{q+1} \mid \cdots \mid \mathbf{a}_n\ ]) = \beta \cdot \det([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{a}_{q-1} \mid \mathbf{b}_q \mid \mathbf{a}_{q+1} \mid \cdots \mid \mathbf{a}_n\ ])$$

2. **Proof of Theorem (β).**

   For each i, denote the i-th entry of $\mathbf{b}_q$ by b_{iq}, and the i-th entry of $\mathbf{c}_q$, by c_{iq}.

   Then the i-th entry of $\mathbf{a}_q$ is given by $a_{iq} = \beta b_{iq} + \gamma c_{iq}$.

   By definition, $A(i|q) = B(i|q) = C(i|q)$ for each i.

   Expand $\det(A)$ along the q-th column:

$$\begin{aligned} & \det(A) \\ = \ & (-1)^{1+q}a_{1q}\det(A(1|q)) + (-1)^{2+q}a_{2q}\det(A(2|q)) + (-1)^{3+q}a_{3q}\det(A(3|q)) + \cdots + (-1)^{n+q}a_{nq}\det(A(n|q)) \\ = \ & (-1)^{1+q}(\beta b_{1q} + \gamma c_{1q})\det(A(1|q)) + (-1)^{2+q}(\beta b_{2q} + \gamma c_{2q})\det(A(2|q)) + (-1)^{3+q}(\beta b_{3q} + \gamma c_{3q})\det(A(3|q)) \\ & + \cdots + (-1)^{n+q}(\beta b_{nq} + \gamma c_{nq})\det(A(n|q)) \\ = \ & \beta[(-1)^{1+q}b_{1q}\det(A(1|q)) + (-1)^{2+q}b_{2q}\det(A(2|q)) + (-1)^{3+q}b_{3q}\det(A(3|q)) + \cdots + (-1)^{n+q}b_{nq}\det(A(n|q))] \\ & + \gamma[(-1)^{1+q}c_{1q}\det(A(1|q)) + (-1)^{2+q}c_{2q}\det(A(2|q)) + (-1)^{3+q}c_{3q}\det(A(3|q)) + \cdots + (-1)^{n+q}c_{nq}\det(A(n|q))] \\ = \ & \beta[(-1)^{1+q}b_{1q}\det(B(1|q)) + (-1)^{2+q}b_{2q}\det(B(2|q)) + (-1)^{3+q}b_{3q}\det(B(3|q)) + \cdots + (-1)^{n+q}b_{nq}\det(B(n|q))] \\ & + \gamma[(-1)^{1+q}c_{1q}\det(C(1|q)) + (-1)^{2+q}c_{2q}\det(C(2|q)) + (-1)^{3+q}c_{3q}\det(C(3|q)) + \cdots + (-1)^{n+q}c_{nq}\det(C(n|q))] \\ = \ & \beta\det(B) + \gamma\det(C) \end{aligned}$$

3. Recall Theorem (α) from the handout *Determinants*:

   Suppose A be a square matrix. Then $\det(A^t) = \det(A)$.

   Combined with Theorem (β), this gives the result below:

   Corollary to Theorem (β). (Multilinearity of determinants in rows.)

   Let R, S, T be $(n \times n)$-square matrix, whose i-th rows are denoted by $\mathbf{r}_i, \mathbf{s}_i, \mathbf{t}_i$ respectively for each i.

   Suppose σ, τ are real numbers, and there is some $p = 1, 2, \cdots, n$ so that:

   (a) $\mathbf{r}_p = \sigma\mathbf{s}_p + \tau\mathbf{t}_p$, *and*

   (b) $\mathbf{r}_i = \mathbf{s}_i = \mathbf{t}_i$ *whenever* $i \neq p$.

   Then $\det(R) = \sigma\det(S) + \tau\det(T)$.

   Remark. What we have obtained is:

$$\det\left(\begin{bmatrix} \mathbf{r}_1 \\ \hline \vdots \\ \hline \mathbf{r}_{p-1} \\ \hline \sigma\mathbf{s}_p + \tau\mathbf{t}_p \\ \hline \mathbf{r}_{p+1} \\ \hline \vdots \\ \hline \mathbf{r}_n \end{bmatrix}\right) = \sigma\det\left(\begin{bmatrix} \mathbf{r}_1 \\ \hline \vdots \\ \hline \mathbf{r}_{p-1} \\ \hline \mathbf{s}_p \\ \hline \mathbf{r}_{p+1} \\ \hline \vdots \\ \hline \mathbf{r}_n \end{bmatrix}\right) + \tau\det\left(\begin{bmatrix} \mathbf{r}_1 \\ \hline \vdots \\ \hline \mathbf{r}_{p-1} \\ \hline \mathbf{t}_p \\ \hline \mathbf{r}_{p+1} \\ \hline \vdots \\ \hline \mathbf{r}_n \end{bmatrix}\right)$$

In particular,

$$\det\left(\begin{bmatrix} \mathbf{r}_1 \\ \hline \vdots \\ \hline \mathbf{r}_{p-1} \\ \hline \sigma\mathbf{s}_p \\ \hline \mathbf{r}_{p+1} \\ \hline \vdots \\ \hline \mathbf{r}_n \end{bmatrix}\right) = \sigma\det\left(\begin{bmatrix} \mathbf{r}_1 \\ \hline \vdots \\ \hline \mathbf{r}_{p-1} \\ \hline \mathbf{s}_p \\ \hline \mathbf{r}_{p+1} \\ \hline \vdots \\ \hline \mathbf{r}_n \end{bmatrix}\right)$$

4. **Lemma (1).**

   Let A, B be $(n \times n)$-square matrix, whose j-th columns are denoted by $\mathbf{a}_j, \mathbf{b}_j$ respectively for each j.

   Suppose there is some $q = 1, 2, \cdots, n$ so that:

   (a) $\mathbf{b}_q = \mathbf{a}_{q+1}$,

   (b) $\mathbf{b}_{q+1} = \mathbf{a}_q$, *and*

   (c) $\mathbf{b}_j = \mathbf{a}_j$ *whenever* $j < q$ *or* $j > q + 1$.

   Then $\det(B) = -\det(A)$.

   Remark. Presented in symbols, what happens is:

$$\det([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{a}_{q-1} \mid \mathbf{a}_{q+1} \mid \mathbf{a}_q \mid \mathbf{a}_{q+2} \mid \cdots \mid \mathbf{a}_n\ ]) = -\det([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{a}_{q-1} \mid \mathbf{a}_q \mid \mathbf{a}_{q+1} \mid \mathbf{a}_{q+2} \mid \cdots \mid \mathbf{a}_n\ ])$$

   In plain words, this results says that the determinant of two square matrices differ by a multiple of -1 when it happens that one of them is resultant from the other by interchanging two neighbouring columns.

5. **Proof of Lemma (1).**

   For each i, denote the i-th entry of $\mathbf{a}_q$ by a_{iq}. Then the i-th entry of $\mathbf{b}_{q+1}$ is given by $b_{i,q+1} = a_{iq}$.

   By definition, $A(i|q) = B(i|q + 1)$ for each i.

   Expand $\det(B)$ along the $(q + 1)$-th column:

$$\begin{aligned}
& \det(B) \\
= \ & (-1)^{1+q+1}b_{1,q+1}\det(B(1|q+1)) + (-1)^{2+q+1}b_{2,q+1}\det(B(2|q+1)) + (-1)^{3+q+1}b_{3,q+1}\det(B(3|q+1)) \\
& + \cdots + (-1)^{n+q+1}b_{n,q+1}\det(B(n|q+1)) \\
= \ & (-1)^{1+q+1}a_{1,q}\det(A(1|q)) + (-1)^{2+q+1}a_{2,q}\det(A(2|q)) + (-1)^{3+q+1}a_{3,q}\det(A(3|q)) \\
& + \cdots + (-1)^{n+q+1}a_{n,q}\det(A(n|q)) \\
= \ & -[(-1)^{1+q}a_{1,q}\det(A(1|q)) + (-1)^{2+q}a_{2,q}\det(A(2|q)) + (-1)^{3+q}a_{3,q}\det(A(3|q)) + \cdots + (-1)^{n+q}a_{n,q}\det(A(n|q))] \\
= \ & -\det(A)
\end{aligned}$$

6. **Theorem (γ).**

   Let A, C be $(n \times n)$-square matrices, whose j-th columns are denoted by $\mathbf{a}_j, \mathbf{c}_j$ respectively for each j.

   Suppose there are some distinct p, q amongst $1, 2, \cdots, n$ so that:

   (a) $\mathbf{c}_q = \mathbf{a}_p$,

   (b) $\mathbf{c}_p = \mathbf{a}_q$, *and*

   (c) $\mathbf{c}_j = \mathbf{a}_j$ *whenever* $j \neq p$ *and* $j \neq q$.

   Then $\det(C) = -\det(A)$.

   Remark. Presented in symbols, what happens is:

$$\det([\ \cdots \mid \mathbf{a}_{p-1} \mid \mathbf{a}_p \mid \mathbf{a}_{p+1} \mid \cdots \mid \mathbf{a}_{q-1} \mid \mathbf{a}_q \mid \mathbf{a}_{q+1} \mid \cdots\ ]) = -\det([\ \cdots \mid \mathbf{a}_{p-1} \mid \mathbf{a}_q \mid \mathbf{a}_{p+1} \mid \cdots \mid \mathbf{a}_{q-1} \mid \mathbf{a}_p \mid \mathbf{a}_{q+1} \mid \cdots\ ])$$

   In plain words, this results says that the determinant of two square matrices differ by a multiple of -1 when it happens that one of them is resultant from the other by interchanging two distinct columns.

   Proof of Theorem (γ). Apply Lemma (1) repeatedly. It takes an odd number of steps of interchanging neighbouring columns to obtain C from A. Each step results in a factor of -1. Hence $\det(C) = -\det(A)$.

7. **Illustration of the idea in the argument for Theorem (γ).**

Suppose $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5 \in \mathbb{R}^5$.
We verify that

$$\det([\ \mathbf{a}_5 \mid \mathbf{a}_2 \mid \mathbf{a}_3 \mid \mathbf{a}_4 \mid \mathbf{a}_1\ ]) = -\det([\ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \mathbf{a}_3 \mid \mathbf{a}_4 \mid \mathbf{a}_5\ ])$$

by repeatedly applying Lemma (1):

$$\begin{aligned}
\det([\ \mathbf{a}_5 \mid \mathbf{a}_2 \mid \mathbf{a}_3 \mid \mathbf{a}_4 \mid \mathbf{a}_1\ ]) &= (-1)\cdot\det([\ \mathbf{a}_2 \mid \mathbf{a}_5 \mid \mathbf{a}_3 \mid \mathbf{a}_4 \mid \mathbf{a}_1\ ]) \\
&= (-1)^2\det([\ \mathbf{a}_2 \mid \mathbf{a}_3 \mid \mathbf{a}_5 \mid \mathbf{a}_4 \mid \mathbf{a}_1\ ]) \\
&= (-1)^3\det([\ \mathbf{a}_2 \mid \mathbf{a}_3 \mid \mathbf{a}_4 \mid \mathbf{a}_5 \mid \mathbf{a}_1\ ]) \\
&= (-1)^4\det([\ \mathbf{a}_2 \mid \mathbf{a}_3 \mid \mathbf{a}_4 \mid \mathbf{a}_1 \mid \mathbf{a}_5\ ]) \\
&= (-1)^5\det([\ \mathbf{a}_2 \mid \mathbf{a}_3 \mid \mathbf{a}_1 \mid \mathbf{a}_4 \mid \mathbf{a}_5\ ]) \\
&= (-1)^6\det([\ \mathbf{a}_2 \mid \mathbf{a}_1 \mid \mathbf{a}_3 \mid \mathbf{a}_4 \mid \mathbf{a}_5\ ]) \\
&= (-1)^7\det([\ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \mathbf{a}_3 \mid \mathbf{a}_4 \mid \mathbf{a}_5\ ]) = -\det([\ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \mathbf{a}_3 \mid \mathbf{a}_4 \mid \mathbf{a}_5\ ])
\end{aligned}$$

8. Two immediate consequences of Theorem (β) and Theorem (γ) are Theorem (δ) and Theorem (ϵ).

Theorem (δ).

The statements below hold:

(a) *Let A be an $(n \times n)$-square matrix.*
Suppose two distinct columns of A are identical. Then $\det(A) = 0$.

(b) *Let A be an $(n \times n)$-square matrix.*
Suppose one column of A is a linear combination of the other columns. Then $\det(A) = 0$.

Remark. From the statement (b), we know that in particular, if:

- one column of A is a scalar multiple of another column, or
- one column of A is a sum of two or more of the other column,

then $\det(A) = 0$.

9. **Proof of Theorem (δ).**

(a) Let A be an $(n \times n)$-square matrix.
Suppose two distinct columns of A, say, the j-th and k-th column, are identical.
Denote by A' the matrix resultant from interchanging these two columns.
By Theorem (γ), $\det(A') = -\det(A)$.
Since the j-th column and the k-th column of A are identical, we have $A = A'$.
Then $\det(A') = \det(A)$.
Since $\det(A') = -\det(A)$ and $\det(A') = \det(A)$, we have $\det(A) = 0$.

(b) Let A be an $(n \times n)$-square matrix, whose j-th column is denoted by $\mathbf{a}_j$.
Without loss of generality, suppose $\mathbf{a}_1$ is a linear combination of $\mathbf{a}_2, \mathbf{a}_3, \cdots, \mathbf{a}_n$.
Then there exist some $\beta_2, \beta_3, \cdots, \beta_n \in \mathbb{R}$ such that $\mathbf{a}_1 = \beta_2\mathbf{a}_2 + \beta_3\mathbf{a}_3 + \cdots + \beta_n\mathbf{a}_n$.
Therefore

$$\begin{aligned}
\det(A) &= \det([\ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \mathbf{a}_3 \mid \cdots \mid \mathbf{a}_n\ ]) \\
&= \det([\ \beta_2\mathbf{a}_2 + \beta_3\mathbf{a}_3 + \cdots + \beta_n\mathbf{a}_n \mid \mathbf{a}_2 \mid \mathbf{a}_3 \mid \cdots \mid \mathbf{a}_n\ ]) \\
&= \beta_2\cdot\det([\ \mathbf{a}_2 \mid \mathbf{a}_2 \mid \mathbf{a}_3 \mid \cdots \mid \mathbf{a}_n\ ]) + \beta_3\cdot\det([\ \mathbf{a}_3 \mid \mathbf{a}_2 \mid \mathbf{a}_3 \mid \cdots \mid \mathbf{a}_n\ ]) \\
&\quad + \cdots + \beta_n\cdot\det([\ \mathbf{a}_n \mid \mathbf{a}_2 \mid \mathbf{a}_3 \mid \cdots \mid \mathbf{a}_n\ ]) \\
&= \beta_2\cdot 0 + \beta_3\cdot 0 + \cdots + \beta_n\cdot 0 = 0
\end{aligned}$$

10. **Theorem (ϵ).**

Let A be an $(n \times n)$-square matrix.

Suppose A' is the $(n \times n)$-square matrix obtained from A by adding a scalar multiple of one column of A to another column of A.

Then $\det(A') = \det(A)$.

Remark. Denote the j-th column of A by $\mathbf{a}_j$ for each j. What this result says is

$$\det([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{a}_i \mid \cdots \mid \alpha\mathbf{a}_i + \mathbf{a}_k \mid \cdots \mid \mathbf{a}_n\ ]) = \det([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{a}_i \mid \cdots \mid \mathbf{a}_k \mid \cdots \mid \mathbf{a}_n\ ])$$

whenever $i \neq k$ and α is a real number.

11. **Proof of Theorem (ϵ).**

Denote the j-th column of A by $\mathbf{a}_j$ for each j. Suppose

$$A' = [\ \mathbf{a}_1 \mid \cdots \mid \mathbf{a}_i \mid \cdots \mid \alpha\mathbf{a}_i + \mathbf{a}_k \mid \cdots \mid \mathbf{a}_n\ ].$$

Then

$$\begin{aligned} \det(A') &= \det([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{a}_i \mid \cdots \mid \alpha\mathbf{a}_i + \mathbf{a}_k \mid \cdots \mid \mathbf{a}_n\ ]) \\ &= \alpha \cdot \det([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{a}_i \mid \cdots \mid \mathbf{a}_i \mid \cdots \mid \mathbf{a}_n\ ]) + 1 \cdot \det([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{a}_i \mid \cdots \mid \mathbf{a}_k \mid \cdots \mid \mathbf{a}_n\ ]) \\ &= \alpha \cdot 0 + \det(A) = \det(A) \end{aligned}$$

12. Again recall Theorem (α) from the handout *Determinants*:

    Suppose A be a square matrix. Then $\det(A^t) = \det(A)$.

13. **Corollary to Theorem (γ).**

Let R, T be $(n \times n)$-square matrices, whose i-th rows are denoted by $\mathbf{r}_i, \mathbf{t}_i$ respectively for each i.

Suppose there are some distinct p, q amongst $1, 2, \cdots, n$ so that:

(a) $\mathbf{t}_q = \mathbf{r}_p$,

(b) $\mathbf{t}_p = \mathbf{r}_q$, *and*

(c) $\mathbf{t}_j = \mathbf{r}_j$ *whenever $j \neq p$ and $j \neq q$.*

Then $\det(T) = -\det(R)$.

Remark. In plain words, this results says that the determinant of two square matrices differ by a multiple of -1 when it happens that one of them is resultant from the other by interchanging two distinct rows:

$$\det\left(\begin{bmatrix} \vdots \\ \hline \mathbf{r}_{p-1} \\ \hline \mathbf{r}_p \\ \hline \vdots \\ \hline \mathbf{r}_{q-1} \\ \hline \mathbf{r}_q \\ \hline \mathbf{r}_{q+1} \\ \hline \vdots \end{bmatrix}\right) = \det\left(\begin{bmatrix} \vdots \\ \hline \mathbf{r}_{p-1} \\ \hline \mathbf{r}_q \\ \hline \vdots \\ \hline \mathbf{r}_{q-1} \\ \hline \mathbf{r}_p \\ \hline \mathbf{r}_{q+1} \\ \hline \vdots \end{bmatrix}\right)$$

14. **Corollary to Theorem (δ).**

The statements below hold:

(a) *Let B be an $(n \times n)$-square matrix.*
    Suppose two distinct rows of B are identical. Then $\det(B) = 0$.

(b) *Let B be an $(n \times n)$-square matrix.*
    Suppose one row of B is a linear combination of the other rows, in the sense that the transpose of that row is a linear combination of the transposes of the other rows. Then $\det(B) = 0$.

Remark. From the statement (b), we know that in particular, if:

- one row of B is a scalar multiple of another row, or
- one row of B is a sum of two or more of the other rows,

then $\det(B) = 0$.

15. **Corollary to Theorem (ϵ).**

Let B be an $(n \times n)$-square matrix.

Suppose B' is the $(n \times n)$-square matrix obtained from A by adding a scalar multiple of one row of B to another row of B.

Then $\det(B') = \det(B)$.

Remark. Denote the i-th row of B by $\mathbf{b}_i$ for each i. What this result says is

$$\det(\begin{bmatrix} \mathbf{b}_1 \\ \hline \vdots \\ \hline \mathbf{b}_j \\ \hline \vdots \\ \hline \beta\mathbf{b}_j + \mathbf{b}_k \\ \hline \vdots \\ \hline \mathbf{b}_n \end{bmatrix}) = \det(\begin{bmatrix} \mathbf{b}_1 \\ \hline \vdots \\ \hline \mathbf{b}_j \\ \hline \vdots \\ \hline \mathbf{b}_k \\ \hline \vdots \\ \hline \mathbf{b}_n \end{bmatrix})$$

whenever $j \neq k$ and β is a real number.

In terms of the language of row operations, that says, when it happens that if B' is obtained from B by the application of the row operation $\alpha R_i + R_k$, then $\det(B') = \det(B)$.

16. **Examples on the applications of Theorem (γ), Theorem (δ), Theorem (ϵ).**

Preparation. We imitate the notations for row operations on matrices to set up notations for column operations on matrices:

- $\alpha C_i + C_k$ reads as 'adding to the k-th column the scalar multiple of the i-th column by α',
- βC_i reads as 'multiplying the i-th column by the (non-zero) number β',
- $C_i \longleftrightarrow C_k$ reads as 'interchanging the i-th column with the k-th column'.

A recurrent theme in these examples is that we always try to apply row/column operations in such a way that more and more 0's will appear in the resultant matrices of the successive applications of the row/column operations.

(a) We have the sequence of row operations

$$\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix} \xrightarrow{-6R_2+R_3} \begin{bmatrix} 1 & 7 & 0 \\ 0 & -33 & 8 \\ 0 & 1 & 5 \end{bmatrix} \xrightarrow{-33R_3+R_2} \begin{bmatrix} 1 & 7 & 0 \\ 0 & 0 & 173 \\ 0 & 1 & 5 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 7 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 173 \end{bmatrix}$$

Correspondingly, we have the equalities

$$\det(\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 7 & 0 \\ 0 & -33 & 8 \\ 0 & 1 & 5 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 7 & 0 \\ 0 & 0 & 173 \\ 0 & 1 & 5 \end{bmatrix}) = -\det(\begin{bmatrix} 1 & 7 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 173 \end{bmatrix}) = -1 \cdot 1 \cdot 173 = -173.$$

(b) We have the sequence of row operations and column operations

$$\begin{bmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ -3 & -1 & 2 \end{bmatrix} \xrightarrow{1R_1+R_3} \begin{bmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-1R_3+R_2} \begin{bmatrix} 3 & 2 & -1 \\ 4 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-2R_3+R_1} \begin{bmatrix} 3 & 0 & -3 \\ 4 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix}$$
$$\xrightarrow{1C_1+C_3} \begin{bmatrix} 3 & 0 & 0 \\ 4 & 0 & 9 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{C_2 \leftrightarrow C_3} \begin{bmatrix} 3 & 0 & 0 \\ 4 & 9 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Correspondingly, we have the equalities

$$\begin{aligned} & \det(\begin{bmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ -3 & -1 & 2 \end{bmatrix}) = \det(\begin{bmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ 0 & 1 & 1 \end{bmatrix}) = \det(\begin{bmatrix} 3 & 2 & -1 \\ 4 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix}) = \det(\begin{bmatrix} 3 & 0 & -3 \\ 4 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix}) \\ = & \det(\begin{bmatrix} 3 & 0 & 0 \\ 4 & 0 & 9 \\ 0 & 1 & 1 \end{bmatrix}) = -\det(\begin{bmatrix} 3 & 0 & 0 \\ 4 & 9 & 0 \\ 0 & 1 & 1 \end{bmatrix}) = -3 \cdot 9 \cdot 1 = -27 \end{aligned}$$

(c) We have the sequence of row operations

$$\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix} \xrightarrow{-1R_3+R_4} \begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{-1R_1+R_3} \begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Correspondingly, we have the equalities

$$\det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 3 \end{bmatrix}) = 1 \cdot 5 \cdot 1 \cdot 3 = 15$$

Alternative method.

We have the sequence of column operations

$$\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix} \xrightarrow{-9C_1+C_2} \begin{bmatrix} 1 & 0 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 0 & 8 & 0 \\ 1 & 0 & 8 & 3 \end{bmatrix} \xrightarrow{-8C_1+C_3} \begin{bmatrix} 1 & 0 & -1 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Hence we have the equalities below due to the above 'column operations' and further due to 'expansion' along third row:

$$\det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 0 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 0 & 8 & 0 \\ 1 & 0 & 8 & 3 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 0 & -1 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix})$$

$$= 1 \cdot \det(\begin{bmatrix} 0 & -1 & 7 \\ 5 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}) = -\det(\begin{bmatrix} 5 & 2 & 5 \\ 0 & -1 & 7 \\ 0 & 0 & 3 \end{bmatrix}) = -5 \cdot (-1) \cdot 3 = 15$$

(d) We have the sequence of row operations and column operations

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix} \xrightarrow{-1R_1+R_5} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{-1R_1+R_4} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\xrightarrow{-1R_1+R_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{-1R_1+R_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{C_1 \leftrightarrow C_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\xrightarrow{C_2 \leftrightarrow C_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Correspondingly, we have the equalities

$$\det(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix})$$

$$= \det(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}) = -\det(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix})$$

$$= \det(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}) = 1 \cdot 1 \cdot 1 \cdot 2 \cdot 3 = 6$$

(e) We have the sequence of row operations and column operations

$$\begin{bmatrix} -2 & 3 & 0 & 1 \\ 9 & -2 & 0 & 1 \\ 1 & 3 & -2 & -1 \\ 4 & 1 & 2 & 6 \end{bmatrix} \xrightarrow{1R_3+R_4} \begin{bmatrix} -2 & 3 & 0 & 1 \\ 9 & -2 & 0 & 1 \\ 1 & 3 & -2 & -1 \\ 5 & 4 & 0 & 5 \end{bmatrix} \xrightarrow{-1R_1+R_2} \begin{bmatrix} -2 & 3 & 0 & 1 \\ 11 & -5 & 0 & 0 \\ 1 & 3 & -2 & -1 \\ 5 & 4 & 0 & 5 \end{bmatrix}$$

$$\xrightarrow{-5R_1+R_4} \begin{bmatrix} -2 & 3 & 0 & 1 \\ 11 & -5 & 0 & 0 \\ 1 & 3 & -2 & -1 \\ 15 & -11 & 0 & 0 \end{bmatrix} \xrightarrow{-3C_4+C_2} \begin{bmatrix} -2 & 0 & 0 & 1 \\ 11 & -5 & 0 & 0 \\ 1 & 6 & -2 & -1 \\ 15 & -11 & 0 & 0 \end{bmatrix} \xrightarrow{-2R_2+R_4} \begin{bmatrix} -2 & 0 & 0 & 1 \\ 11 & -5 & 0 & 0 \\ 1 & 6 & -2 & -1 \\ -7 & -1 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-5R_4+R_2} \begin{bmatrix} -2 & 0 & 0 & 1 \\ 46 & 0 & 0 & 0 \\ 1 & 6 & -2 & -1 \\ -7 & -1 & 0 & 0 \end{bmatrix}$$

Hence we have the equalities

$$\det\left(\begin{bmatrix} -2 & 3 & 0 & 1 \\ 9 & -2 & 0 & 1 \\ 1 & 3 & -2 & -1 \\ 4 & 1 & 2 & 6 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -2 & 3 & 0 & 1 \\ 9 & -2 & 0 & 1 \\ 1 & 3 & -2 & -1 \\ 5 & 4 & 0 & 5 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -2 & 3 & 0 & 1 \\ 11 & -5 & 0 & 0 \\ 1 & 3 & -2 & -1 \\ 5 & 4 & 0 & 5 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} -2 & 3 & 0 & 1 \\ 11 & -5 & 0 & 0 \\ 1 & 3 & -2 & -1 \\ 15 & -11 & 0 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -2 & 0 & 0 & 1 \\ 11 & -5 & 0 & 0 \\ 1 & 6 & -2 & -1 \\ 15 & -11 & 0 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -2 & 0 & 0 & 1 \\ 11 & -5 & 0 & 0 \\ 1 & 6 & -2 & -1 \\ -7 & -1 & 0 & 0 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} -2 & 0 & 0 & 1 \\ 46 & 0 & 0 & 0 \\ 1 & 6 & -2 & -1 \\ -7 & -1 & 0 & 0 \end{bmatrix}\right) = -46\det\left(\begin{bmatrix} 0 & 0 & 1 \\ 6 & -2 & -1 \\ -1 & 0 & 0 \end{bmatrix}\right) = (-46)(-2)\det\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = 92$$

(f) We have the sequence of row operations and column operations

$$\begin{bmatrix} 2 & 0 & 2 & 3 \\ 1 & 3 & -1 & 1 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{bmatrix} \xrightarrow{-1C_1+C_3} \begin{bmatrix} 2 & 0 & 0 & 3 \\ 1 & 3 & -2 & 1 \\ -1 & 1 & 0 & 2 \\ 3 & 5 & 1 & 0 \end{bmatrix} \xrightarrow{-3R_3+R_2} \begin{bmatrix} 2 & 0 & 0 & 3 \\ 4 & 0 & -2 & -5 \\ -1 & 1 & 0 & 2 \\ 3 & 5 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{2C_3+C_1} \begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 0 & -2 & -11 \\ -1 & 1 & 0 & 2 \\ 3 & 5 & 1 & 0 \end{bmatrix} \xrightarrow{1C_2+C_1} \begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 0 & -2 & -11 \\ 0 & 1 & 0 & 2 \\ 8 & 5 & 1 & 0 \end{bmatrix} \xrightarrow{-4R_1+R_4} \begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 0 & -2 & -11 \\ 0 & 1 & 0 & 2 \\ 0 & 5 & 1 & -12 \end{bmatrix}$$

$$\xrightarrow{-5R_3+R_4} \begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 0 & -2 & -11 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -22 \end{bmatrix} \xrightarrow{2R_4+R_2} \begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & -55 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -22 \end{bmatrix}$$

Hence we have the equalities

$$\det\left(\begin{bmatrix} 2 & 0 & 2 & 3 \\ 1 & 3 & -1 & 1 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2 & 0 & 0 & 3 \\ 1 & 3 & -2 & 1 \\ -1 & 1 & 0 & 2 \\ 3 & 5 & 1 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2 & 0 & 0 & 3 \\ 4 & 0 & -2 & -5 \\ -1 & 1 & 0 & 2 \\ 3 & 5 & 1 & 0 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 0 & -2 & -11 \\ -1 & 1 & 0 & 2 \\ 3 & 5 & 1 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 0 & -2 & -11 \\ 0 & 1 & 0 & 2 \\ 8 & 5 & 1 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 0 & -2 & -11 \\ 0 & 1 & 0 & 2 \\ 0 & 5 & 1 & -12 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 0 & -2 & -11 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -22 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & -55 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -22 \end{bmatrix}\right)$$

$$= 2\det\left(\begin{bmatrix} 0 & 0 & -55 \\ 1 & 0 & 2 \\ 0 & 1 & -22 \end{bmatrix}\right) = 2(-55)\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = -110$$