

1. Definition. (Submatrices of a square matrix)

Let A be an $(n \times n)$ -square matrix.

For each k, ℓ , the (k, ℓ) -th submatrix of A is defined to be the $((n-1) \times (n-1))$ -matrix resultant from simultaneously deleted the k -th row and ℓ -th column of A . It is denoted by $A(k|\ell)$.

2. Illustration.

(a) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Then $A(1|1) = [a_{22}]$, $A(1|2) = [a_{21}]$, $A(2|1) = [a_{12}]$, $A(2|2) = [a_{11}]$.

(b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Then

$$\begin{aligned} A(1|1) &= \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, & A(1|2) &= \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, & A(1|3) &= \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \\ A(2|1) &= \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}, & A(2|2) &= \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}, & A(2|3) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}, \\ A(3|1) &= \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}, & A(3|2) &= \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}, & A(3|3) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{aligned}$$

3. Definition. ('Inductive definition for determinants' through 'expansion' along the first column.)

Let A be an $(n \times n)$ -square matrix, whose (i, j) -th entry is denoted by a_{ij} .

(a) Suppose $n = 1$. Then we define the determinant of A , which is denoted by $\det(A)$, to be the number which is the only entry of A .

(b) Suppose $n > 1$. Then we define the determinant of A , which is denoted by $\det(A)$, by

$$\det(A) = (-1)^{1+1}a_{11} \det(A(1|1)) + (-1)^{2+1}a_{21} \det(A(2|1)) + (-1)^{3+1}a_{31} \det(A(3|1)) + \dots + (-1)^{n+1}a_{n1} \det(A(n|1)).$$

Remark. The 'formula'

$$\det(A) = (-1)^{1+1}a_{11} \det(A(1|1)) + (-1)^{2+1}a_{21} \det(A(2|1)) + (-1)^{3+1}a_{31} \det(A(3|1)) + \dots + (-1)^{n+1}a_{n1} \det(A(n|1))$$

is usually referred to as the 'expansion' of a determinant along the first column.

4. Illustration.

(a) Suppose $A = [a_{11}]$. Then $\det(A) = a_{11}$.

(b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. We have $A(1|1) = [a_{22}]$ and $A(2|1) = [a_{12}]$.

Then $\det(A) = a_{11} \det(A(1|1)) - a_{21} \det(A(2|1)) = a_{11}a_{22} - a_{12}a_{21}$.

(c) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

We have $A(1|1) = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$, $A(2|1) = \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$, and $A(3|1) = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$.

Then

$$\begin{aligned} \det(A) &= a_{11} \det(A(1|1)) - a_{21} \det(A(2|1)) + a_{31} \det(A(3|1)) \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

(d) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$.

We have $A(1|1) = \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}$, $A(2|1) = \begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}$, $A(3|1) = \begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}$, and

$A(4|1) = \begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{bmatrix}$.

Then

$$\begin{aligned}
 \det(A) &= a_{11} \det(A(1|1)) - a_{21} \det(A(2|1)) + a_{31} \det(A(3|1)) - a_{41} \det(A(4|1)) \\
 &= \dots \\
 &= a_{11}a_{22}a_{33}a_{44} + a_{11}a_{23}a_{34}a_{42} + a_{11}a_{24}a_{32}a_{43} \\
 &\quad + a_{12}a_{21}a_{34}a_{43} + a_{12}a_{24}a_{33}a_{41} + a_{12}a_{23}a_{31}a_{44} \\
 &\quad + a_{13}a_{24}a_{31}a_{42} + a_{13}a_{21}a_{32}a_{44} + a_{13}a_{22}a_{34}a_{41} \\
 &\quad + a_{14}a_{23}a_{32}a_{41} + a_{14}a_{22}a_{31}a_{43} + a_{14}a_{21}a_{33}a_{42} \\
 &\quad - a_{11}a_{22}a_{34}a_{43} - a_{11}a_{24}a_{33}a_{42} - a_{11}a_{23}a_{32}a_{44} \\
 &\quad - a_{12}a_{21}a_{33}a_{44} - a_{12}a_{23}a_{34}a_{41} - a_{12}a_{24}a_{31}a_{43} \\
 &\quad - a_{13}a_{24}a_{32}a_{41} - a_{13}a_{22}a_{31}a_{44} - a_{13}a_{21}a_{34}a_{42} \\
 &\quad - a_{14}a_{23}a_{31}a_{42} - a_{14}a_{21}a_{32}a_{43} - a_{14}a_{22}a_{33}a_{41}
 \end{aligned}$$

5. Examples.

(a) $\det\left(\begin{bmatrix} 1 & 7 \\ 6 & 9 \end{bmatrix}\right) = 1 \cdot 9 - 6 \cdot 7 = -33.$

(b) $\det\left(\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}\right) = 1 \cdot \det\left(\begin{bmatrix} 9 & 8 \\ 1 & 5 \end{bmatrix}\right) - 6 \cdot \det\left(\begin{bmatrix} 7 & 0 \\ 1 & 5 \end{bmatrix}\right) + 0 \cdot \det\left(\begin{bmatrix} 7 & 0 \\ 9 & 8 \end{bmatrix}\right) = 1 \cdot (9 \cdot 5 - 1 \cdot 8) - 6(7 \cdot 5 - 1 \cdot 0) = -173.$

(c)

$$\begin{aligned}
 &\det\left(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}\right) \\
 &= 1 \cdot \det\left(\begin{bmatrix} 5 & 2 & 5 \\ 9 & 8 & 0 \\ 9 & 8 & 3 \end{bmatrix}\right) - 0 \cdot \det\left(\begin{bmatrix} 9 & 7 & 7 \\ 9 & 8 & 0 \\ 9 & 8 & 3 \end{bmatrix}\right) + 1 \cdot \det\left(\begin{bmatrix} 9 & 7 & 7 \\ 5 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix}\right) - 1 \cdot \det\left(\begin{bmatrix} 9 & 7 & 7 \\ 5 & 2 & 5 \\ 9 & 8 & 0 \end{bmatrix}\right) \\
 &= \left(5 \cdot \det\left(\begin{bmatrix} 8 & 0 \\ 8 & 3 \end{bmatrix}\right) - 9 \cdot \det\left(\begin{bmatrix} 2 & 5 \\ 8 & 3 \end{bmatrix}\right) + 9 \cdot \det\left(\begin{bmatrix} 2 & 5 \\ 8 & 0 \end{bmatrix}\right)\right) - 0 \\
 &\quad + \left(9 \cdot \det\left(\begin{bmatrix} 2 & 5 \\ 8 & 3 \end{bmatrix}\right) - 5 \cdot \det\left(\begin{bmatrix} 7 & 7 \\ 8 & 3 \end{bmatrix}\right) + 9 \cdot \det\left(\begin{bmatrix} 7 & 7 \\ 2 & 5 \end{bmatrix}\right)\right) \\
 &\quad - \left(9 \cdot \det\left(\begin{bmatrix} 2 & 5 \\ 8 & 0 \end{bmatrix}\right) - 5 \cdot \det\left(\begin{bmatrix} 7 & 7 \\ 8 & 0 \end{bmatrix}\right) + 9 \cdot \det\left(\begin{bmatrix} 7 & 7 \\ 2 & 5 \end{bmatrix}\right)\right) \\
 &= [5 \cdot 24 - 9 \cdot (-34) + 9 \cdot (-40)] - 0 + [9 \cdot (-34) - 5 \cdot (-35) + 9 \cdot 21] - [9 \cdot (-40) - 5 \cdot (-56) + 9 \cdot (21)] \\
 &= 15
 \end{aligned}$$

6. Theorem (1). ('Expansion' of a determinant along any arbitrary column.)

Let A be an $(n \times n)$ -square matrix, whose (i, j) -th entry is denoted by a_{ij} .

Suppose $n > 1$. Then, for each $j = 1, 2, \dots, n$,

$$\det(A) = (-1)^{1+j} a_{1j} \det(A(1|j)) + (-1)^{2+j} a_{2j} \det(A(2|j)) + (-1)^{3+j} a_{3j} \det(A(3|j)) + \dots + (-1)^{n+j} a_{nj} \det(A(n|j)).$$

Proof of Theorem (1). Omitted. (This can be done with mathematical induction.)

7. Illustration.

(a) Suppose A is a (3×3) -square matrix, whose (i, j) -th entry is denoted by a_{ij} . Then

$$\begin{aligned}
 \det(A) &= a_{11} \det(A(1|1)) - a_{21} \det(A(2|1)) + a_{31} \det(A(3|1)), \\
 \det(A) &= -a_{12} \det(A(1|2)) + a_{22} \det(A(2|2)) - a_{32} \det(A(3|2)), \\
 \det(A) &= a_{13} \det(A(1|3)) - a_{23} \det(A(2|3)) + a_{33} \det(A(3|3)).
 \end{aligned}$$

(b) Suppose A is a (4×4) -square matrix, whose (i, j) -th entry is denoted by a_{ij} . Then

$$\begin{aligned}
 \det(A) &= a_{11} \det(A(1|1)) - a_{21} \det(A(2|1)) + a_{31} \det(A(3|1)) - a_{41} \det(A(4|1)), \\
 \det(A) &= -a_{12} \det(A(1|2)) + a_{22} \det(A(2|2)) - a_{32} \det(A(3|2)) + a_{42} \det(A(4|2)), \\
 \det(A) &= a_{13} \det(A(1|3)) - a_{23} \det(A(2|3)) + a_{33} \det(A(3|3)) - a_{43} \det(A(4|3)), \\
 \det(A) &= -a_{14} \det(A(1|4)) + a_{24} \det(A(2|4)) - a_{34} \det(A(3|4)) + a_{44} \det(A(4|4)).
 \end{aligned}$$

8. Examples.

$$(a) \det\left(\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}\right) = -7 \cdot \det\left(\begin{bmatrix} 6 & 8 \\ 0 & 5 \end{bmatrix}\right) + 9 \cdot \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\right) - 1 \cdot \det\left(\begin{bmatrix} 1 & 0 \\ 6 & 8 \end{bmatrix}\right) = \dots = -173.$$

$$\det\left(\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}\right) = 0 \cdot \det\left(\begin{bmatrix} 6 & 9 \\ 0 & 1 \end{bmatrix}\right) - 8 \cdot \det\left(\begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}\right) + 5 \cdot \det\left(\begin{bmatrix} 1 & 7 \\ 6 & 9 \end{bmatrix}\right) = \dots = -173.$$

(b)

$$\det\left(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}\right) \\ = -9 \cdot \det\left(\begin{bmatrix} 0 & 2 & 5 \\ 1 & 8 & 0 \\ 1 & 8 & 3 \end{bmatrix}\right) + 5 \cdot \det\left(\begin{bmatrix} 1 & 7 & 7 \\ 1 & 8 & 0 \\ 1 & 8 & 3 \end{bmatrix}\right) - 9 \cdot \det\left(\begin{bmatrix} 1 & 7 & 7 \\ 0 & 2 & 5 \\ 1 & 8 & 3 \end{bmatrix}\right) + 9 \cdot \det\left(\begin{bmatrix} 1 & 7 & 7 \\ 0 & 2 & 5 \\ 1 & 8 & 0 \end{bmatrix}\right) = \dots = 15$$

$$\det\left(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}\right) \\ = 7 \cdot \det\left(\begin{bmatrix} 0 & 5 & 5 \\ 1 & 9 & 0 \\ 1 & 9 & 3 \end{bmatrix}\right) - 2 \cdot \det\left(\begin{bmatrix} 1 & 9 & 7 \\ 1 & 9 & 0 \\ 1 & 9 & 3 \end{bmatrix}\right) + 8 \cdot \det\left(\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 5 \\ 1 & 9 & 3 \end{bmatrix}\right) - 8 \cdot \det\left(\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 5 \\ 1 & 9 & 0 \end{bmatrix}\right) = \dots = 15$$

$$\det\left(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}\right) \\ = -7 \cdot \det\left(\begin{bmatrix} 0 & 5 & 2 \\ 1 & 9 & 8 \\ 1 & 9 & 8 \end{bmatrix}\right) + 5 \cdot \det\left(\begin{bmatrix} 1 & 9 & 7 \\ 1 & 9 & 8 \\ 1 & 9 & 8 \end{bmatrix}\right) - 0 \cdot \det\left(\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 2 \\ 1 & 9 & 8 \end{bmatrix}\right) + 3 \cdot \det\left(\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 2 \\ 1 & 9 & 8 \end{bmatrix}\right) = \dots = 15$$

9. Theorem (2). ('Expansion' of a determinant along the first row.)

Let A be an $(n \times n)$ -square matrix, whose (i, j) -th entry is denoted by a_{ij} .

Suppose $n > 1$. Then

$$\det(A) = (-1)^{1+1}a_{11} \det(A(1|1)) + (-1)^{1+2}a_{12} \det(A(1|2)) + (-1)^{1+3}a_{13} \det(A(1|3)) + \dots + (-1)^{1+n}a_{1n} \det(A(1|n)).$$

Proof of Theorem (2). Omitted. (This can be done with mathematical induction.)

10. Illustration.

(a) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. We have $A(1|1) = [a_{22}]$ and $A(2|1) = [a_{12}]$.

$$\text{Then } \det(A) = a_{11} \det(A(1|1)) - a_{12} \det(A(2|1)) = a_{11}a_{22} - a_{12}a_{21}.$$

(b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

$$\text{We have } A(1|1) = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, A(1|2) = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \text{ and } A(1|3) = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

Then

$$\begin{aligned} \det(A) &= a_{11} \det(A(1|1)) - a_{12} \det(A(1|2)) + a_{13} \det(A(1|3)) \\ &= \dots \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

(c) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$.

$$\text{We have } A(1|1) = \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}, A(1|2) = \begin{bmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{bmatrix}, A(1|3) = \begin{bmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{bmatrix}, \text{ and}$$

$$A(1|4) = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}.$$

Then

$$\begin{aligned}
\det(A) &= a_{11} \det(A(1|1)) - a_{12} \det(A(1|2)) + a_{13} \det(A(1|3)) - a_{14} \det(A(1|4)) \\
&= \dots \\
&= a_{11}a_{22}a_{33}a_{44} + a_{11}a_{23}a_{34}a_{42} + a_{11}a_{24}a_{32}a_{43} \\
&\quad + a_{12}a_{21}a_{34}a_{43} + a_{12}a_{24}a_{33}a_{41} + a_{12}a_{23}a_{31}a_{44} \\
&\quad + a_{13}a_{24}a_{31}a_{42} + a_{13}a_{21}a_{32}a_{44} + a_{13}a_{22}a_{34}a_{41} \\
&\quad + a_{14}a_{23}a_{32}a_{41} + a_{14}a_{22}a_{31}a_{43} + a_{14}a_{21}a_{33}a_{42} \\
&\quad - a_{11}a_{22}a_{34}a_{43} - a_{11}a_{24}a_{33}a_{42} - a_{11}a_{23}a_{32}a_{44} \\
&\quad - a_{12}a_{21}a_{33}a_{44} - a_{12}a_{23}a_{34}a_{41} - a_{12}a_{24}a_{31}a_{43} \\
&\quad - a_{13}a_{24}a_{32}a_{41} - a_{13}a_{22}a_{31}a_{44} - a_{13}a_{21}a_{34}a_{42} \\
&\quad - a_{14}a_{23}a_{31}a_{42} - a_{14}a_{21}a_{32}a_{43} - a_{14}a_{22}a_{33}a_{41}
\end{aligned}$$

11. **Theorem (α).**

Suppose A be a square matrix. Then $\det(A^t) = \det(A)$.

Proof of Theorem (α). Omitted. (This can be done with mathematical induction. Apply the definition and Theorem (2).)

12. **Theorem (3). ('Expansion' of a determinant along any arbitrary row.)**

Let A be an $(n \times n)$ -square matrix, whose (i, j) -th entry is denoted by a_{ij} .

Suppose $n > 1$. Then, for each $i = 1, 2, \dots, n$,

$$\det(A) = (-1)^{i+1}a_{i1} \det(A(i|1)) + (-1)^{i+2}a_{i2} \det(A(i|2)) + (-1)^{i+3}a_{i3} \det(A(i|3)) + \dots + (-1)^{i+n}a_{in} \det(A(i|n)).$$

Proof of Theorem (3). This is a consequence of Theorem (1) and Theorem (α) combined.

13. **Illustration.**

(a) Suppose A is a (3×3) -square matrix, whose (i, j) -th entry is denoted by a_{ij} . Then

$$\begin{aligned}
\det(A) &= a_{11} \det(A(1|1)) - a_{12} \det(A(1|2)) + a_{13} \det(A(1|3)), \\
\det(A) &= -a_{21} \det(A(2|1)) + a_{22} \det(A(2|2)) - a_{23} \det(A(2|3)), \\
\det(A) &= a_{31} \det(A(3|1)) - a_{32} \det(A(3|2)) + a_{33} \det(A(3|3)).
\end{aligned}$$

(b) Suppose A is a (4×4) -square matrix, whose (i, j) -th entry is denoted by a_{ij} . Then

$$\begin{aligned}
\det(A) &= a_{11} \det(A(1|1)) - a_{12} \det(A(1|2)) + a_{13} \det(A(1|3)) - a_{14} \det(A(1|4)), \\
\det(A) &= -a_{21} \det(A(2|1)) + a_{22} \det(A(2|2)) - a_{23} \det(A(2|3)) + a_{24} \det(A(2|4)), \\
\det(A) &= a_{31} \det(A(3|1)) - a_{32} \det(A(3|2)) + a_{33} \det(A(3|3)) - a_{34} \det(A(3|4)), \\
\det(A) &= -a_{41} \det(A(4|1)) + a_{42} \det(A(4|2)) - a_{43} \det(A(4|3)) + a_{44} \det(A(4|4)).
\end{aligned}$$

14. **Key theoretical examples.**

(a) Let A be an $(n \times n)$ -square matrix, whose (i, j) -th entry is denoted by a_{ij} .

i. Suppose there is some q amongst $1, 2, \dots, n$ for which the q -th column is all zero.

Then

$$\begin{aligned}
\det(A) &= (-1)^{1+q}a_{1q} \det(A(1|q)) + (-1)^{2+q}a_{2q} \det(A(2|q)) + (-1)^{3+q}a_{3q} \det(A(3|q)) \\
&\quad + \dots + (-1)^{n+q}a_{nq} \det(A(n|q)) \\
&= 0.
\end{aligned}$$

ii. Suppose there is some p amongst $1, 2, \dots, n$ for which the p -th row is all zero.

Then

$$\begin{aligned}
\det(A) &= (-1)^{p+1}a_{p1} \det(A(p|1)) + (-1)^{p+2}a_{p2} \det(A(p|2)) + (-1)^{p+3}a_{p3} \det(A(p|3)) \\
&\quad + \dots + (-1)^{p+n}a_{pn} \det(A(p|n)) \\
&= 0.
\end{aligned}$$

(b) Let D be an $(n \times n)$ -square matrix, whose (i, j) -th entry is denoted by d_{ij} .

Suppose $d_{ij} = 0$ whenever $i \neq j$. (Such a matrix is called a diagonal matrix.)

We have $\det(D) = d_{11} \det(D(1|1))$.

Note that $D(1|1)$ is itself a diagonal matrix.

Then $\det(D) = d_{11}d_{22}d_{33} \dots d_{nn}$.

Remark. In particular, $\det(I_n) = 1$.

- (c) i. Let A be an $(n \times n)$ -square matrix, whose (i, j) -th entry is denoted by a_{ij} .
Suppose $a_{ij} = 0$ whenever $i > j$. (Such a matrix is called an upper-triangular matrix.)

$$A \text{ is explicitly given by } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a_{nn} \end{bmatrix}.$$

By definition, we have $\det(A) = a_{11} \det(A(1|1))$.

$$\text{Note that } A(1|1) = \begin{bmatrix} a_{22} & a_{23} & a_{24} & \cdots & \cdots & a_{2n} \\ 0 & a_{33} & a_{34} & \cdots & \cdots & a_{3n} \\ 0 & 0 & a_{44} & \cdots & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a_{nn} \end{bmatrix}. \text{ (So } A(1|1) \text{ is also an upper triangular matrix.)}$$

Then, 'inductively', we have $\det(A) = a_{11}a_{22}a_{33} \cdots a_{nn}$.

- ii. Let B be an $(n \times n)$ -square matrix, whose (i, j) -th entry is denoted by b_{ij} .
Suppose $b_{ij} = 0$ whenever $i < j$. (Such a matrix is called a lower-triangular matrix.)

$$B \text{ is explicitly given by } B = \begin{bmatrix} b_{11} & 0 & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & 0 & 0 & \cdots & 0 \\ b_{31} & b_{32} & b_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & \cdots & b_{nn} \end{bmatrix}.$$

Note that B^t is an upper triangular matrix.

Then $\det(B) = \det(B^t) = b_{11}b_{22}b_{33} \cdots b_{nn}$.

- (d) i. Denote by ρ the row operation $\alpha R_i + R_k$ on matrices with n rows.
The row operation matrix $M[\rho]$ corresponding to ρ is the $(n \times n)$ -square matrix given by $M[\rho] = I_n + \alpha E_{k,i}^{n,n}$.
Note that $M[\rho]$ is an upper triangular matrix or a lower triangular matrix.
Then $\det(M[\rho]) = 1$.
- ii. Denote by σ the row operation βR_i on matrices with n rows.
The row operation matrix $M[\sigma]$ corresponding to σ is given by $M[\sigma] = I_n + (\beta - 1)E_{i,i}^{n,n}$.
Note that $M[\sigma]$ is an upper triangular matrix, whose diagonal entries are made of $n - 1$ copies of 1 and 1 copy of β .
Then $\det(M[\sigma]) = \beta$.
- iii. Denote by τ the row operation $R_i \leftrightarrow R_k$ on matrices with n rows.
The row operation matrix $M[\tau]$ corresponding to τ is given by $M[\tau] = I_n - E_{i,i}^{n,n} - E_{k,k}^{n,n} + E_{k,i}^{n,n} + E_{i,k}^{n,n}$.
 $M[\tau]$ is a symmetric matrix with $n - 2$ entries of 1, along the diagonal, and 2 entries of 1 off diagonal.
Repeatedly applying the definition, we have $\det(M[\tau]) = \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = -1$.
- (e) Let A be an $(n \times n)$ -square matrix. Suppose A is a reduced row-echelon form.
- i. Suppose A is non-singular. Then $A = I_n$. Therefore $\det(A) = 1$.
 - ii. Suppose A is singular. Then A has at least one entire row of 0's. Therefore $\det(A) = 0$.