1. **Definition. (Diagonal matrix.)**

   Let D be a $(n \times n)$-square matrix, whose (i,j)-th entry is denoted by d_{ij}. The matrix D is said to be a diagonal matrix if and only if $d_{ij} = 0$ whenever $i \neq j$.

   Remark on notation. When $d_{11} = \alpha_1, d_{22} = \alpha_2, \cdots, d_{nn} = \alpha_n$, we may write $D = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_n)$.

2. **Definition. (Diagonalizability and diagonalization.)**

   Let A be an $(n \times n)$-square matrix.

   (a) *Suppose U is a non-singular $(n \times n)$-square matrix. Then we say that $U^{-1}AU$ is a diagonalization of A if and only if $U^{-1}AU$ is a diagonal matrix.*

   (b) *A is said to be diagonalizable if and only if there is some non-singular $(n \times n)$-square matrix T such that $T^{-1}AT$ is a diagonalization of A.*

   Remark. A diagonalizable matrix may have various diagonalization.

3. Recall the definition for the notions of *eigenvalue* and *eigenvector* from the handout *Eigenvalues and eigenvectors*:

   Let A be an $(n \times n)$-square matrix (with real entries). Let λ be a (real) number. Let $\mathbf{v}$ be a non-zero vector with n (real) entries.

   We say $\mathbf{v}$ is an eigenvector of A with eigenvalue λ (or equivalently, λ is an eigenvalue of A with a corresponding eigenvector $\mathbf{v}$) if and only if $A\mathbf{v} = \lambda\mathbf{v}$.

4. **Theorem (C).**

   Let A is an $(n \times n)$-square matrix. Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^n$, and $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n\ ]$.

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathbb{R}^n$. (So U is non-singular.)

   Then the statements below are logically equivalent:

   (a) *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are eigenvectors of A, with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ respectively.*

   (b) *$U^{-1}AU$ is a diagonal matrix, given by $U^{-1}AU = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.*

5. **Corollary to Theorem (C).**

   Let A is an $(n \times n)$-square matrix.

   Suppose A has n pairwise distinct eigenvalues. Then A is diagonalizable.

   Proof of Corollary to Theorem (C).

   Each of the n eigenvalues of A will correspond to an eigenvector. Since the eigenvalues are pairwise distinct, the n corresponding eigenvectors will be linearly independent. These n vectors will then constitute a basis for $\mathbb{R}^n$.

6. **Examples of diagonalizable matrices and their diagonalizations.**

   (a) Let $A = \begin{bmatrix} 13 & 30 \\ -6 & -14 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

   It happens that $\mathbf{u}_1, \mathbf{u}_2$ are eigenvectors of A with respective eigenvalues $1, -2$.

   Since $\mathbf{u}_1, \mathbf{u}_2$ are eigenvectors of A with distinct eigenvalues, they are linearly independent.

   Then $\mathbf{u}_1, \mathbf{u}_2$ constitute a basis for $\mathbb{R}^2$.

   Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2\ ]$.

   U is nonsingular, and $U^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix}$.

   By direct verification, we see that $U^{-1}AU = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, as expected from theory.

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of A with respective eigenvalues $1, 2, 3$.

Since $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of A with pairwise distinct eigenvalues, they are linear independent.

Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ constitute a basis for $\mathbb{R}^3$.

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ]$.

U is nonsingular, and $U^{-1} = \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1/2 \end{bmatrix}$.

By direct verification, we see that $U^{-1}AU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, as expected from theory.

(c) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of A with respective eigenvalues $4, 1, 1$.

We can check that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linear independent. (Fill in the detail.)

Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ constitute a basis for $\mathbb{R}^3$.

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ]$.

U is nonsingular, and $U^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{bmatrix}$.

By direct verification, we see that $U^{-1}AU = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, as expected from theory.

(d) Let $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ -5 & 2 & 5 & -1 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 5 \\ -1 \\ -5 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -5 \\ -3 \\ 15 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are eigenvectors of A with respective eigenvalues $1, -1, 3, -3$.

Since $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are eigenvectors of A with pairwise distinct eigenvalues, they are linear independent.

Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ constitute a basis for $\mathbb{R}^4$.

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4\ ]$.

U is nonsingular, and $U^{-1} = \begin{bmatrix} 5/8 & -1/4 & 0 & -1/8 \\ 1/4 & 1/8 & -1/8 & 0 \\ 0 & 1/8 & 5/24 & 1/12 \\ 1/8 & 0 & -1/12 & 1/24 \end{bmatrix}$.

By direct verification, we see that $U^{-1}AU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$, as expected from theory.

7. **Non-examples on diagonalizability.**

(a) Let b be a real number, and $A = \begin{bmatrix} b & 1 & 0 \\ 0 & b & 1 \\ 0 & 0 & b \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

u is an eigenvector of A with eigenvalue b, and every eigenvector of A is a non-zero scalar multiple of $\mathbf{u}$.

Then there is no basis for $\mathbb{R}^3$ which is constituted by eigenvectors of A.

Therefore A is not diagonalizable.

(b) Let $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

A has no eigenvalues, and hence no eigenvectors.

Then there is no basis for $\mathbb{R}^4$ which is constituted by eigenvectors of A.

Therefore A is not diagonalizable.

8. **Proof of Theorem (C).**

Let A is an $(n \times n)$-square matrix. Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^n$, and $U = \left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n\ \right]$.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathbb{R}^n$.

- Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are eigenvectors of A, with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ respectively.

  [Reminder: We want to verify that a diagonalization for A is given by $U^{-1}AU = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.]

  Then for each $j = 1, 2, \cdots, n$, we have $A\mathbf{u}_j = \lambda_j \mathbf{u}_j$.

  Therefore

$$\begin{aligned} AU &= \left[\ A\mathbf{u}_1 \mid A\mathbf{u}_2 \mid \cdots \mid A\mathbf{u}_n\ \right] = \left[\ \lambda_1\mathbf{u}_1 \mid \lambda_2\mathbf{u}_2 \mid \cdots \mid \lambda_n\mathbf{u}_n\ \right] \\ &= \left[\ \lambda_1 U\mathbf{e}_1^{(n)} \mid \lambda_2 U\mathbf{e}_2^{(n)} \mid \cdots \mid \lambda_n U\mathbf{e}_n^{(n)}\ \right] = \left[\ U(\lambda_1\mathbf{e}_1^{(n)}) \mid U(\lambda_2\mathbf{e}_2^{(n)}) \mid \cdots \mid U(\lambda_n\mathbf{e}_n^{(n)})\ \right] \\ &= U\left[\ \lambda_1\mathbf{e}_1^{(n)} \mid \lambda_2\mathbf{e}_2^{(n)} \mid \cdots \mid \lambda_n\mathbf{e}_n^{(n)}\ \right] = U\,\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) \end{aligned}$$

  Since $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitutes a basis for $\mathbb{R}^n$, U is non-singular and invertible. The matrix U^{-1} is well-defined.

  Then $U^{-1}AU = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, which is a diagonal matrix.

- Suppose $U^{-1}AU$ is a diagonal matrix, given by $U^{-1}AU = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

  [Reminder: We want to verify that for each j, $\mathbf{u}_j$ is an eigenvector of A with eigenvalue λ_j.]

  Then

$$\begin{aligned} \left[\ A\mathbf{u}_1 \mid A\mathbf{u}_2 \mid \cdots \mid A\mathbf{u}_n\ \right] &= AU = U\,\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) \\ &= U\left[\ \lambda_1\mathbf{e}_1^{(n)} \mid \lambda_2\mathbf{e}_2^{(n)} \mid \cdots \mid \lambda_n\mathbf{e}_n^{(n)}\ \right] \\ &= \left[\ U(\lambda_1\mathbf{e}_1^{(n)}) \mid U(\lambda_2\mathbf{e}_2^{(n)}) \mid \cdots \mid U(\lambda_n\mathbf{e}_n^{(n)})\ \right] \\ &= \left[\ \lambda_1 U\mathbf{e}_1^{(n)} \mid \lambda_2 U\mathbf{e}_2^{(n)} \mid \cdots \mid \lambda_n U\mathbf{e}_n^{(n)}\ \right] \\ &= \left[\ \lambda_1\mathbf{u}_1 \mid \lambda_2\mathbf{u}_2 \mid \cdots \mid \lambda_n\mathbf{u}_n\ \right] \end{aligned}$$

  Therefore, for each $j = 1, 2, \cdots, n$, we have $A\mathbf{u}_j = \lambda_j\mathbf{u}_j$.

  Hence $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are eigenvectors of A, with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ respectively.

9. **Lemma (1).**

Suppose A is an $(n \times n)$-square matrix. Then A is singular if and only if 0 is an eigenvalue of A.

Furthermore, if A is singular then every non-zero vector in $\mathcal{N}(A)$ is an eigenvector of A with eigenvalue 0.

Remark. When $\dim(\mathcal{N}(A)) \geq 2$, we do not expect any two arbitrary non-zero vectors in $\mathcal{N}(A)$ to be scalar multiples of each other. This result reminds us that we should not expect eigenvectors corresponding to the same eigenvalue of A to be non-zero scalar multiples of each other.

10. **Lemma (2).**

Let A be an $(n \times n)$-square matrix. Suppose λ is a real number. Then the statements below hold:

(a) *λ is an eigenvalue of A if and only if $A - \lambda I_n$ is singular.*

(b) *Now suppose λ is an eigenvalue of A indeed. Then for any non-zero vector $\mathbf{x}$ in $\mathbb{R}^n$, $\mathbf{x}$ is an eigenvector of A with eigenvalue λ if and only if $\mathbf{x} \in \mathcal{N}(A - \lambda I_n)$.*

11. **Definition. (Eigenspace.)**

Let A be an $(n \times n)$-square matrix. Suppose λ be an eigenvalue of A. Then $\mathcal{N}(A - \lambda I_n)$ is called the eigenspace of A with eigenvalue λ. It is denoted by $\mathcal{E}_A(\lambda)$.

The dimension of $\mathcal{E}_A(\lambda)$ is called the geometric multiplicity of the eigenvalue λ of A.

12. **Theorem (D).**

Let A is an $(n \times n)$-square matrix. Suppose $\lambda_1, \lambda_2, \cdots, \lambda_p$ are all the eigenvalues of A, pairwise distinct.

Then the statements below are logically equivalent:

(a) *A is diagonalizable.*

(b) *The sum of the respective geometric multiplicities of $\lambda_1, \lambda_2, \cdots, \lambda_p$, as eigenvalues of A, equals n.*

Now suppose A is indeed diagaonalizable.

For each $k = 1, 2, \cdots, p$, write $\dim(\mathcal{E}_A(\lambda_k)) = n_k$, and suppose that $\mathbf{v}_{k,1}, \mathbf{v}_{k,2}, \cdots, \mathbf{v}_{k,n_k}$ constitute a basis for $\mathcal{E}_A(\lambda_k)$. Then a basis for $\mathbb{R}^n$ is constituted by

$$\begin{array}{llll} \mathbf{v}_{1,1}, \mathbf{v}_{1,2}, \cdots, \mathbf{v}_{1,n_1}, & & & \\ & \mathbf{v}_{2,1}, \mathbf{v}_{2,2}, \cdots, \mathbf{v}_{2,n_2}, & & \\ & & \ddots & \\ & & & \mathbf{v}_{p,1}, \mathbf{v}_{p,2}, \cdots, \mathbf{v}_{p,n_p}. \end{array}$$

Proof. Omitted. (The argument is not difficult at a conceptual level; we can certainly give it within the context of this course. However it will be tedious unless we introduce the notion of *direct sum*.)

13. **Illustration of the content of Theorem (D).**

(a) Let $A = \begin{bmatrix} 13 & 30 \\ -6 & -14 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2$ are eigenvectors of A with respective eigenvalues $1, -2$.

The only eigenspaces of A are $\mathcal{E}_A(1), \mathcal{E}_A(-2)$.

The dimension of $\mathcal{E}_A(1)$ is 1, with a basis given by $\mathbf{u}_1$.

The dimension of $\mathcal{E}_A(-2)$ is 1, with a basis given by $\mathbf{u}_2$.

Since $\dim(\mathcal{E}_A(1)) + \dim(\mathcal{E}_A(-2)) = 2 = \dim(\mathbb{R}^2)$, A is expected to be diagonalizable.

A diagonalization of A is given by $U^{-1}AU = \mathrm{diag}(1, -2)$, in which $U = \begin{bmatrix} \mathbf{u}_1 \mid \mathbf{u}_2 \end{bmatrix}$.

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of A with respective eigenvalues $1, 2, 3$.

The only eigenspaces of A are $\mathcal{E}_A(1), \mathcal{E}_A(2), \mathcal{E}_A(3)$.

The dimension of $\mathcal{E}_A(1)$ is 1, with a basis given by $\mathbf{u}_1$.

The dimension of $\mathcal{E}_A(2)$ is 1, with a basis given by $\mathbf{u}_2$.

The dimension of $\mathcal{E}_A(3)$ is 1, with a basis given by $\mathbf{u}_3$.

Since $\dim(\mathcal{E}_A(1)) + \dim(\mathcal{E}_A(2)) + \dim(\mathcal{E}_A(3)) = 3 = \dim(\mathbb{R}^3)$, A is expected to be diagonalizable.

A diagonalization of A is given by $U^{-1}AU = \mathrm{diag}(1, 2, 3)$, in which $U = \begin{bmatrix} \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \end{bmatrix}$.

(c) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of A with respective eigenvalues $4, 1, 1$.

The only eigenspaces of A are $\mathcal{E}_A(4), \mathcal{E}_A(1)$.

The dimension of $\mathcal{E}_A(4)$ is 1, with a basis given by $\mathbf{u}_1$.

The dimension of $\mathcal{E}_A(1)$ is 2, with a basis given by $\mathbf{u}_2, \mathbf{u}_3$.

Since $\dim(\mathcal{E}_A(4)) + \dim(\mathcal{E}_A(1)) = 3 = \dim(\mathbb{R}^3)$, A is expected to be diagonalizable.

A diagonalization of A is given by $U^{-1}AU = \mathrm{diag}(4, 1, 1)$, in which $U = \begin{bmatrix} \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \end{bmatrix}$.

(d) Let $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ -5 & 2 & 5 & -1 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 5 \\ -1 \\ -5 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -5 \\ -3 \\ 15 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are eigenvectors of A with respective eigenvalues $1, -1, 3, -3$.

The only eigenspaces of A are $\mathcal{E}_A(1), \mathcal{E}_A(-1), \mathcal{E}_A(3), \mathcal{E}_A(-3)$.

The dimension of $\mathcal{E}_A(1)$ is 1, with a basis given by $\mathbf{u}_1$.

The dimension of $\mathcal{E}_A(-1)$ is 1, with a basis given by $\mathbf{u}_2$.

The dimension of $\mathcal{E}_A(3)$ is 1, with a basis given by $\mathbf{u}_3$.

The dimension of $\mathcal{E}_A(-3)$ is 1, with a basis given by $\mathbf{u}_4$.

Since $\dim(\mathcal{E}_A(1)) + \dim(\mathcal{E}_A(-1)) + \dim(\mathcal{E}_A(3)) + \dim(\mathcal{E}_A(-3)) = 4 = \dim(\mathbb{R}^4)$, A is expected to be diagonalizable.

A diagonalization of A is given by $U^{-1}AU = \mathrm{diag}(1, -1, 3, -3)$, in which $U = \left[\begin{array}{c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{array}\right]$.

(e) Let b be a real number, and $A = \begin{bmatrix} b & 1 & 0 \\ 0 & b & 1 \\ 0 & 0 & b \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

b is the only eigenvalue of A, and every eigenvector of A is a non-zero scalar multiple of $\mathbf{u}$.

The only eigenspace of A is $\mathcal{E}_A(b)$, which is of dimension 1.

Therefore A is not diagonalizable.

(f) Let $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

A has no eigenvalues, and hence no eigenspace.

Therefore A is not diagonalizable.

14. **Theorem (3).**

Let A be an $(n \times n)$-square matrix. Suppose A is diagonalizable, with a diagonalization $U^{-1}AU = D$, for some non-singular $(n \times n)$-square matrix U and for some $(n \times n)$-diagonal matrix D.

Then the statements below hold:

(a) *For each positive integer p, A^p is diagonalizable, with a diagonalization given by $U^{-1}A^pU = D^p$.*

(b) *Suppose A is non-singular. Then D is non-singular, and A^{-1} is diagonalizable, with a diagonalization given by $U^{-1}A^{-1}U = D^{-1}$.*

Proof of Theorem (3). Exercise.

Remark. This result tells us that when A is diagonalizable, it will be easy to find its positive powers. (Why? Because it is easy to find the positive powers of a diagonal matrix.)

15. **Examples on application of Theorem (3).**

(a) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

It happens that A is diagonalizable, with a diagonalization given by $U^{-1}AU = \mathrm{diag}(1, 2, 3)$, in which $U = \left[\begin{array}{c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{array}\right]$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$.

Then $A = U\,\mathrm{diag}(1, 2, 3)U^{-1}$.

Note that $U = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ and $U^{-1} = \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1/2 \end{bmatrix}$.

Then for each positive integer p, we have

$$\begin{aligned} A^p &= U(\mathrm{diag}(1,2,3))^p U^{-1} = U(\mathrm{diag}(1, 2^p, 3^p))U^{-1} \\ &= \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^p & 0 \\ 0 & 0 & 3^p \end{bmatrix} \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1+2^p & 1/2 - 2 \cdot 2^p + (3/2) \cdot 3^p \\ 0 & 2^p & -2 \cdot 2^p + 2 \cdot 3^p \\ 0 & 0 & 3^p \end{bmatrix}. \end{aligned}$$

(b) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

It happens that A is diagonalizable, with a diagonalization given by $U^{-1}AU = \mathrm{diag}(4,1,1)$, in which $U = \left[\begin{array}{c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{array}\right]$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Then $A = U\,\mathrm{diag}(4,1,1)U^{-1}$.

Note that $U = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ and $U^{-1} = \dfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$.

Then for each positive integer p, we have

$$\begin{aligned} A^p &= U(\mathrm{diag}(4,1,1))^p U^{-1} = U(\mathrm{diag}(4^p,1,1))U^{-1} \\ &= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 4^p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 4^p & 4^p-3 & 4^p+3 \\ 4^p-1 & 4^p+2 & 4^p-1 \\ 4^p+1 & 4^p+1 & 4^p-2 \end{bmatrix}. \end{aligned}$$