MATH1030 Eigenvalues and eigenvectors.

1. **Definition. (Eigenvalues and eigenvectors.)**

   Let A be an $(n \times n)$-square matrix (with real entries). Let λ be a (real) number. Let $\mathbf{v}$ be a non-zero vector with n (real) entries.

   We say $\mathbf{v}$ is an eigenvector of A with eigenvalue λ (or equivalently, λ is an eigenvalue of A with a corresponding eigenvector $\mathbf{v}$) if and only if $A\mathbf{v} = \lambda\mathbf{v}$.

   Remark. In this course, we restrict all discussion to real numbers. However, in other mathematics courses and in many practical situations, allowing complex numbers into play is not only desirable but also natural and necessary. This definition is formulated in such a way that it can be adapted to the 'world of complex numbers' immediately: simply change the word 'real' to 'complex'.

   Further remarks on terminologies.

   (a) We may write '*the number λ is an eigenvalue of A*' (without mentioning any specific corresponding eigenvector) if there is some non-zero vector $\mathbf{u}$ so that $\mathbf{u}$ is an eigenvector of A with eigenvalue λ.

   (b) We may write '*the non-zero vector $\mathbf{v}$ is an eigenvector of A*' (without mentioning its corresponding eigenvalue) if the equality $A\mathbf{v} = \mu\mathbf{v}$ holds for some number μ.

2. **Lemma (1).**

   Let A be an $(n \times n)$-square matrix. The statements below hold:

   (a) *Let $\mathbf{v}$ be a non-zero vector in $\mathbb{R}^n$. Let λ, μ be real numbers. Suppose $\mathbf{v}$ is an eigenvector of A with eigenvalue λ and also with eigenvalue μ. Then $\lambda = \mu$.*

   (b) *Let $\mathbf{v}$ be a non-zero vector in $\mathbb{R}^n$. Let λ be a real number. Suppose $\mathbf{v}$ is an eigenvector of A with eigenvalue λ. Then, for any non-zero real number β, the vector $\beta\mathbf{v}$ is an eigenvector of A with eigenvalue λ.*

   (c) *Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_\mathbf{k}$ be non-zero vectors in $\mathbb{R}^n$. Let λ be a real number. Suppose each of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ is an eigenvector of A with eigenvalue λ. Then, for any real numbers $\alpha_1, \alpha_2, \cdots, \alpha_k$, if $\alpha_1\mathbf{u}_1 + \alpha\mathbf{u}_2 + \cdots + \alpha\mathbf{u}_k \neq \mathbf{0}$, then the vector $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k$ is an eigenvector of A with eigenvalue λ.*

   Remark. In plain words, what this result says is about the square matrix A:

   (a) Every eigenvector of A corresponds to a unique eigenvalue.

   (b) Every non-zero scalar multiple of an eigenvector of A is also an eigenvector of A, and they correspond to the same eigenvalue.

   (Be careful: we have not ruled out the possibility that non-zero vectors which are not scalar multiple of each other can be eigenvectors of A with the same eigenvalue.)

   (c) When it is not the zero vector, a linear combination of eigenvectors of A with the same eigenvalue is an eigenvector of A with that eigenvalue.

3. **Examples.**

   (a) Let $A = \begin{bmatrix} 13 & 30 \\ -6 & -14 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

   Note that none of $\mathbf{u}, \mathbf{v}$ is the zero vector in $\mathbb{R}^2$.

   i. We have $A\mathbf{u} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} = 1 \cdot \mathbf{u}$. Then $\mathbf{u}$ is an eigenvector of A with eigenvalue 1.

   Every non-zero scalar multiple of $\mathbf{u}$ is also an eigenvector of A with eigenvalue 1. Detail:
   - Suppose $\alpha \in \mathbb{R}$ and $\alpha \neq 0$. Then $A(\alpha\mathbf{u}) = \alpha A\mathbf{u} = \alpha \cdot 1\mathbf{u} = 1 \cdot \alpha\mathbf{u}$.

   ii. We have $A\mathbf{v} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} = -2\mathbf{v}$. Then $\mathbf{v}$ is an eigenvector of A with eigenvalue -2.

   Every non-zero scalar multiple of $\mathbf{v}$ is also an eigenvector of A with eigenvalue -2. Detail:
   - Suppose $\alpha \in \mathbb{R}$ and $\alpha \neq 0$. Then $A(\alpha\mathbf{v}) = \alpha A\mathbf{v} = \alpha(-2\mathbf{v}) = -2\alpha\mathbf{v}$.

   (b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$.

   Note that none of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is the zero vector in $\mathbb{R}^3$.

i. We have $A\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot \mathbf{u}$. Then $\mathbf{u}$ is an eigenvector of A with eigenvalue 1.

Every non-zero scalar multiple of $\mathbf{u}$ is also an eigenvector of A with eigenvalue 1. Detail:

- Suppose $\alpha \in \mathbb{R}$ and $\alpha \neq 0$. Then $A(\alpha\mathbf{u}) = \alpha A\mathbf{u} = \alpha \cdot 1\mathbf{u} = 1 \cdot \alpha\mathbf{u}$.

ii. We have $A\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = 2\mathbf{v}$. Then $\mathbf{v}$ is an eigenvector of A with eigenvalue 2.

Every non-zero scalar multiple of $\mathbf{v}$ is also an eigenvector of A with eigenvalue 2. Detail:

- Suppose $\alpha \in \mathbb{R}$ and $\alpha \neq 0$. Then $A(\alpha\mathbf{v}) = \alpha A\mathbf{v} = \alpha \cdot 2\mathbf{v} = 2 \cdot \alpha\mathbf{v}$.

iii. We have $A\mathbf{w} = \begin{bmatrix} 9 \\ 12 \\ 6 \end{bmatrix} = 3\mathbf{w}$. Then $\mathbf{w}$ is an eigenvector of A with eigenvalue 3.

Every non-zero scalar multiple of $\mathbf{w}$ is also an eigenvector of A with eigenvalue 3. Detail:

- Suppose $\alpha \in \mathbb{R}$ and $\alpha \neq 0$. Then $A(\alpha\mathbf{w}) = \alpha A\mathbf{w} = \alpha \cdot 3\mathbf{w} = 3 \cdot \alpha\mathbf{w}$.

(c) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Note that none of $\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2$ is the zero vector in $\mathbb{R}^3$.

i. We have $A\mathbf{u} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = 4\mathbf{u}$. Then $\mathbf{u}$ is an eigenvector of A with eigenvalue 4.

Every non-zero scalar multiple of $\mathbf{u}$ is also an eigenvector of A with eigenvalue 4.

ii. We have $A\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1 \cdot \mathbf{v}_1$. Then $\mathbf{v}_1$ is an eigenvector of A with eigenvalue 1.

We have $A\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 \cdot \mathbf{v}_2$. Then $\mathbf{v}_2$ is an eigenvector of A with eigenvalue 1.

Every linear combination of $\mathbf{v}_1, \mathbf{v}_2$ which is not the zero vector in $\mathbb{R}^3$ is also an eigenvector of A with eigenvalue 1. Detail:

- Suppose $\alpha_1, \alpha_2 \in \mathbb{R}$, and $\mathbf{w} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$. Suppose $\mathbf{w} \neq \mathbf{0}_3$.
  Then $A\mathbf{w} = A(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2) = \alpha_1 A\mathbf{v}_1 + \alpha_2 A\mathbf{v}_2 = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 = 1 \cdot \mathbf{w}$.

Remark. It is possible for a square matrix to have eigenvectors which are not scalar multiples of each other but which correspond to the same eigenvalue.
More generally, it is possible for a square matrix to have several linearly independent eigenvectors with the same eigenvalue.

(d) Let $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ -5 & 2 & 5 & -1 \end{bmatrix}$, and $\mathbf{t} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 5 \\ -1 \\ -5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 \\ -5 \\ -3 \\ 15 \end{bmatrix}$.

Note that none of $\mathbf{t}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ is the zero vector in $\mathbb{R}^4$.

i. We have $A\mathbf{t} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 1 \cdot \mathbf{t}$. Then $\mathbf{t}$ is an eigenvector of A with eigenvalue 1.

ii. We have $A\mathbf{u} = \begin{bmatrix} -1 \\ -5 \\ 1 \\ 5 \end{bmatrix} = -1 \cdot \mathbf{u}$. Then $\mathbf{u}$ is an eigenvector of A with eigenvalue -1.

iii. We have $A\mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ 9 \\ 9 \end{bmatrix} = 3\mathbf{v}$. Then $\mathbf{v}$ is an eigenvector of A with eigenvalue 3.

iv. We have $A\mathbf{w} = \begin{bmatrix} -3 \\ 15 \\ 9 \\ -45 \end{bmatrix} = -3\mathbf{w}$. Then $\mathbf{w}$ is an eigenvector of A with eigenvalue -3.

4. **Further examples and non-examples.**

(a) Let $A = \mathcal{O}_{n\times n}$.
For any $\mathbf{v} \in \mathbb{R}^n$, $A\mathbf{v} = \mathbf{0}_n = 0 \cdot \mathbf{v}$.
It follows that every non-zero vector in $\mathbb{R}^n$ is an eigenvector of A with eigenvalue 0.

(b) Let $A = I_n$.

For any $\mathbf{v} \in \mathbb{R}^n$, $A\mathbf{v} = \mathbf{v} = 1 \cdot \mathbf{v}$.

It follows that every non-zero vector in $\mathbb{R}^n$ is an eigenvector of A with eigenvalue 1.

(c) Let $b_1, b_2, b_3 \in \mathbb{R}$, and $A = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}$.

For each $j = 1, 2, 3$, we have $A\mathbf{e}_j^{(3)} = b_j \mathbf{e}_j^{(3)}$.

It follows that $\mathbf{e}_j^{(3)}$ is an eigenvector of A with eigenvalue b_j.

(d) Let b be a real number, and $A = \begin{bmatrix} b & 1 & 0 \\ 0 & b & 1 \\ 0 & 0 & b \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

We have $A\mathbf{u} = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} = b\mathbf{u}$.

Then $\mathbf{u}$ is an eigenvector of A with eigenvalue b.

We verify that b is the only eigenvalue of A and every eigenvector of A is a scalar multiple of $\mathbf{u}$:

- Suppose $\mathbf{v}$ is an eigenvector of A with eigenvalue λ. Denote the j-th entry of $\mathbf{v}$ by v_j for each $j = 1, 2, 3$.
  Then $\lambda\mathbf{v} = A\mathbf{v} = \begin{bmatrix} bv_1 + v_2 \\ bv_2 + v_3 \\ bv_3 \end{bmatrix} = b\mathbf{v} + \begin{bmatrix} v_2 \\ v_3 \\ 0 \end{bmatrix}$.
  Therefore $\begin{bmatrix} v_2 \\ v_3 \\ 0 \end{bmatrix} = (\lambda - b)\mathbf{v} = (\lambda - b) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.
  Comparing the entries, we have $(\lambda - b)v_3 = 0$, and $v_2 = (\lambda - b)v_1$ and $v_3 = (\lambda - b)v_2$.
  We have $\lambda = b$ or $v_3 = 0$.
  We claim that $\lambda = b$:
  - ∗ Suppose it were true that $\lambda \neq b$. Then $\lambda - b \neq 0$ and $v_3 = 0$. So $v_2 = \dfrac{v_3}{\lambda - b} = 0$ and $v_1 = \dfrac{v_2}{\lambda - b} = 0$.
    Now $\mathbf{v} = \mathbf{0}$. But this is impossible because $\mathbf{v}$ is an eigenvector of A.

  Therefore $\lambda = b$. Then $v_2 = v_3 = 0$. Hence $\mathbf{v} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = v_1\mathbf{u}$.

Remark. It is possible for an $(n \times n)$-square matrix to have 'relatively few' eigenvectors, in the sense that there is no chance to form a basis for $\mathbb{R}^n$ out of eigenvectors.

(e) Let $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

We verify that A has no eigenvalue (and no eigenvector):

- Suppose $\mathbf{u}$ is an eigenvector of A with eigenvalue λ.
  Denote the j-th entry of $\mathbf{u}$ by u_j.
  We have $\lambda\mathbf{u} = A\mathbf{u} = \begin{bmatrix} u_1 - u_4 \\ u_1 + u_2 \\ u_2 + u_3 \\ u_3 + u_4 \end{bmatrix} = \mathbf{u} + \begin{bmatrix} -u_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}$.
  Then $(\lambda - 1) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = (\lambda - 1)\mathbf{u} = \begin{bmatrix} -u_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}$.
  Comparing the entries, we have
  $$\begin{cases} (\lambda - 1)u_1 &= -u_4 \\ (\lambda - 1)u_2 &= u_1 \\ (\lambda - 1)u_3 &= u_2 \\ (\lambda - 1)u_4 &= u_3 \end{cases}$$
  Then $u_1 = -(\lambda - 1)^4 u_1$. Therefore $[(\lambda - 1)^4 + 1]u_1 = 0$.
  We have $(\lambda - 1)^4 + 1 = 0$ or $u_1 = 0$.
  Since λ is a real number, $(\lambda - 1)^4 + 1 > 0$. Then $u_1 = 0$.
  Therefore $u_2 = u_3 = u_4 = 0$ also.
  Hence $\mathbf{u} = \mathbf{0}$. But this is impossible, because $\mathbf{u}$ is an eigenvector of A.

Remark. It is possible for a square matrix to have no eigenvalue (and hence to have no eigenvector).

5. **Lemma (2).**

   Let A be an $(n \times n)$-square matrix.

   Suppose $\mathbf{u}, \mathbf{v}$ are eigenvectors of A with distinct eigenvalues. Then $\mathbf{u}, \mathbf{v}$ are linearly independent.

6. **Lemma (3).**

   Let A be an $(n \times n)$-square matrix.

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are linearly independent eigenvectors of A.

   Further suppose $\mathbf{v}$ is an eigenvector of A whose eigenvalue is distinct from the eigenvalue of each of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$.

   Then $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k, \mathbf{v}$ are linearly independent.

7. **Question.**

   What are the immediate consequences of Lemma (2) and Lemma (3) combined?

   Answer.
   Suppose A is an $(n \times n)$-square matrix, and $\lambda_1, \lambda_2, \lambda_3, \cdots$ are pairwise distinct eigenvalues of A.

   (a) For each j, suppose $\mathbf{v}_j$ is an eigenvector of A with eigenvalue λ_j.
   By Lemma (2), $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent.
   Then by Lemma (3), $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.
   Again by Lemma (3), $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly independent.
   Repeatedly by Lemma (3), $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent for each k.
   It follows that any finitely many vectors from amongst $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots$ are linearly independent.

   (b) Now, for each j, we indeed pick some eigenvector of A, say, $\mathbf{v}_j$, with eigenvalue λ_j.
   As argued above, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots$ are linearly independent.
   Further recall that any $n+1$ vectors in $\mathbb{R}^n$ are linearly dependent.
   Then the list $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots$ has to terminate somewhere: it is $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$, for some $p \leq n$, in disguise.
   This in turn implies that the p pairwise distinct numbers $\lambda_1, \lambda_2, \cdots, \lambda_p$ are all the eigenvalues that A has.

   We summarize the above discussion into Theorem (A) and Theorem (B) below.

8. **Theorem (A).**

   Suppose A is a square matrix, and $\lambda_1, \lambda_2, \lambda_3, \cdots$ are pairwise distinct eigenvalues.

   Further suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots$ are eigenvectors of A with respective eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots$.

   Then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots$ are linearly independent.

   Theorem (B).

   Suppose A is an $(n \times n)$-square matrix. Then A has at most n pairwise distinct eigenvalues.

   Remark. What we will be most interested is the scenario in which A is an $(n \times n)$-square matrix and there is some basis for $\mathbb{R}^n$ which are eigenvectors of A. (Their eigenvalues are not necessarily pairwise distinct.)

9. **Proof of Lemma (2).**

   Let A be an $(n \times n)$-square matrix.

   Suppose $\mathbf{u}, \mathbf{v}$ are eigenvectors of A with distinct eigenvalues, say, μ, λ respectively.

   [Reminder: we want to deduce that $\mathbf{u}, \mathbf{v}$ are linearly independent.

   This amounts to verifying: '*For any $\alpha, \beta \in \mathbb{R}$, if $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}$ then $\alpha = \beta = 0$.*']

   Pick any $\alpha, \beta \in \mathbb{R}$. Suppose $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}$.

   > [Reminder: We want to deduce $\alpha = \beta = 0$.
   > Ask: How to make use of assumption? Answer: Multiply this equality by A from the left.]

   By assumption, $A\mathbf{u} = \mu\mathbf{u}$ and $A\mathbf{v} = \lambda\mathbf{v}$.

   Then $\alpha\mu\mathbf{u} + \beta\lambda\mathbf{v} = \alpha A\mathbf{u} + \beta A\mathbf{v} = A(\alpha\mathbf{u} + \beta\mathbf{v}) = A\mathbf{0} = \mathbf{0}$.

   We also have $\alpha\lambda\mathbf{u} + \beta\lambda\mathbf{v} = \lambda(\alpha\mathbf{u} + \beta\mathbf{v}) = \mathbf{0}$.

   Then $\alpha(\mu - \lambda)\mathbf{u} = \mathbf{0}$.

   By assumption, $\mathbf{u}$ is not the zero vector. Also by assumption, $\mu - \lambda \neq 0$. Then $\alpha = 0$.

   Now we have $\beta\mathbf{v} = \mathbf{0}$. By assumption, $\mathbf{v}$ is not the zero vector. Then $\beta = 0$.

10. **Proof of Lemma (3).**

Let A be an $(n \times n)$-square matrix.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are linearly independent eigenvectors of A, say, with eigenvalues $\mu_1, \mu_2, \cdots, \mu_k$ respectively.

Further suppose $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, and $\lambda \neq \mu_1$, $\lambda \neq \mu_2$, ..., $\lambda \neq \mu_k$.

[Reminder: we want to deduce that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k, \mathbf{v}$ are linearly independent.

This amounts to verifying: '*For any* $\alpha_1, \alpha_2, \cdots, \alpha_k, \beta \in \mathbb{R}$, *if* $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k + \beta\mathbf{v} = \mathbf{0}$ *then* $\alpha_1 = \alpha_2 = \cdots = \alpha_k = \beta = 0$.']

Pick any $\alpha_1, \alpha_2, \cdots, \alpha_k, \beta \in \mathbb{R}$. Suppose $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k + \beta\mathbf{v} = \mathbf{0}$.

[Reminder: We want to deduce $\alpha_1 = \alpha_2 = \cdots = \alpha_k = \beta = 0$.
Ask: How to make use of assumption? Answer: Multiply this equality by A from the left.]

By assumption, $A\mathbf{u}_j = \mu_j\mathbf{u}_j$ for each $j = 1, 2, \cdots, k$, and $A\mathbf{v} = \lambda\mathbf{v}$. Then

$$\begin{aligned} \alpha_1\mu_1\mathbf{u}_1 + \alpha_2\mu_2\mathbf{u}_2 + \cdots + \alpha_k\mu_k\mathbf{u}_k + \beta\lambda\mathbf{v} &= \alpha_1 A\mathbf{u}_1 + \alpha_2 A\mathbf{u}_2 + \cdots + \alpha_k A\mathbf{u}_k + \beta A\mathbf{v} \\ &= A(\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k + \beta\mathbf{v}) \\ &= A\mathbf{0} = \mathbf{0} \end{aligned}$$

We also have $\alpha_1\lambda\mathbf{u}_1 + \alpha_2\lambda\mathbf{u}_2 + \cdots + \alpha_k\lambda\mathbf{u}_k + \beta\lambda\mathbf{v} = \lambda(\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k + \beta\mathbf{v}) = \mathbf{0}$.

Then $\alpha_1(\mu_1 - \lambda)\mathbf{u}_1 + \alpha_2(\mu_2 - \lambda)\mathbf{u}_2 + \cdots + \alpha_k(\mu_k - \lambda)\mathbf{u}_k = \mathbf{0}$.

By assumption, $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are linearly independent.

Then for each j, we have $\alpha_j(\mu_j - \lambda) = 0$. By assumption, $\lambda \neq \mu_j$. Then $\alpha_j = 0$.

Now we have $\beta\mathbf{v} = 0$. By assumption $\mathbf{v}$ is not the zero vector. Then $\beta = 0$.

It follows that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k, \mathbf{v}$ are linearly independent.

11. **Examples: Applications of Theorem (A) and Theorem (B).**

(a) Let $A = \begin{bmatrix} 13 & 30 \\ -6 & -14 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

It is known that $\mathbf{u}$ is an eigenvector of A with eigenvalue 1, and $\mathbf{v}$ is an eigenvector of A with eigenvalue -2.

According to Theorem (B), A cannot have any other eigenvalues.

One question remains: Can A have any eigenvector which is neither a non-zero scalar multiple of $\mathbf{u}$ nor a non-zero scalar multiple of $\mathbf{v}$?

We show that this cannot happen:

i. Since $\mathbf{u}, \mathbf{v}$ are eigenvectors of A of distinct eigenvalues, they are linearly independent vectors. Since they are vectors in $\mathbb{R}^2$, they constitute a basis for $\mathbb{R}^2$

ii. Suppose $\mathbf{w}$ is an eigenvector of A with eigenvalue μ.
Since $\mathbf{w}$ is a vector in $\mathbb{R}^2$, $\mathbf{w}$ is a linear combination of $\mathbf{u}, \mathbf{v}$.
Then there exist some $\alpha, \beta \in \mathbb{R}$ such that $\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$.

iii. Recall that the only eigenvalues of A are $1, -2$.
Then $\mu = 1$ or $\mu = -2$.

- (Case 1.) Suppose $\mu = 1$.
Then $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{w} = A\mathbf{w} = A(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha A\mathbf{u} + \beta A\mathbf{v} = \alpha\mathbf{u} - 2\beta\mathbf{v}$.
Therefore $3\beta\mathbf{v} = \mathbf{0}$. Since $\mathbf{v} \neq \mathbf{0}$, $\beta = 0$.
Then $\mathbf{w} = \alpha\mathbf{u}$.
Therefore $\mathbf{w}$ is a scalar multiple of $\mathbf{u}$.
- (Case 2.) Suppose $\mu = -2$.
Then $-2\alpha\mathbf{u} - 2\beta\mathbf{v} = -2\mathbf{w} = A\mathbf{w} = A(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha A\mathbf{u} + \beta A\mathbf{v} = \alpha\mathbf{u} - 2\beta\mathbf{v}$.
Therefore $3\alpha\mathbf{u} = \mathbf{0}$. Since $\mathbf{u} \neq \mathbf{0}$, $\alpha = 0$.
Then $\mathbf{w} = \beta\mathbf{v}$.
Therefore $\mathbf{w}$ is a scalar multiple of $\mathbf{v}$.

Hence, in any case, $\mathbf{w}$ is a scalar multiple of $\mathbf{u}$ or a scalar multiple of $\mathbf{v}$.

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$.

It is known that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are eigenvectors of A with eigenvalues $1, 2, 3$ respectively.

According to Theorem (B), A cannot have any other eigenvalues.

One question remains: Can A have any eigenvector which is neither a non-zero scalar multiple of $\mathbf{u}$, nor a non-zero scalar multiple of $\mathbf{v}$, nor a non-zero scalar multiple of $\mathbf{w}$?

We show that this cannot happen:

i. Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are eigenvectors of A of distinct eigenvalues, they are linearly independent vectors. Since they are vectors in $\mathbb{R}^3$, they constitute a basis for $\mathbb{R}^3$

ii. Suppose $\mathbf{t}$ is an eigenvector of A with eigenvalue μ.
Since $\mathbf{t}$ is a vector in $\mathbb{R}^3$, $\mathbf{t}$ is a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$.
Then there exist some $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\mathbf{t} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$.

iii. Recall that the only eigenvalues of A are $1, 2, 3$.
Then $\mu = 1$ or $\mu = 2$ or $\mu = 3$.

- (Case 1.) Suppose $\mu = 1$.
  Then $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} = \mathbf{t} = A\mathbf{t} = A(\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}) = \alpha A\mathbf{u} + \beta A\mathbf{v} + \gamma A\mathbf{w} = \alpha\mathbf{u} + 2\beta\mathbf{v} + 3\gamma\mathbf{w}$.
  Therefore $2\beta\mathbf{v} + 3\gamma\mathbf{w} = \mathbf{0}$. Since $\mathbf{v}, \mathbf{w}$ are linearly independent, $\beta = \gamma = 0$.
  Then $\mathbf{t} = \alpha\mathbf{u}$.
  Therefore $\mathbf{t}$ is a scalar multiple of $\mathbf{u}$.
- (Case 2.) Suppose $\mu = 2$.
  Modifying the above argument, we deduce that $\alpha = \gamma = 0$ and $\mathbf{t} = \beta\mathbf{v}$.
  Therefore $\mathbf{t}$ is a scalar multiple of $\mathbf{v}$.
- (Case 3.) Suppose $\mu = 3$.
  Modifying the above argument, we deduce that $\alpha = \beta = 0$ and $\mathbf{t} = \gamma\mathbf{w}$.
  Therefore $\mathbf{t}$ is a scalar multiple of $\mathbf{w}$.

Hence, in any case, $\mathbf{t}$ is a scalar multiple of $\mathbf{u}$ or a scalar multiple of $\mathbf{v}$ or a scalar multiple of $\mathbf{w}$.

(c) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

It is known that $\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2$ are eigenvectors of A with eigenvalues $4, 1, 1$ respectively.

Observe that $\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2$ constitute a basis for $\mathbb{R}^3$.

We are going to show that A cannot have any other eigenvalue, and that every eigenvector of A is a non-zero scalar multiple of $\mathbf{u}$ or a non-zero vector which is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$.

i. Suppose $\mathbf{w}$ is an eigenvector of A with eigenvalue μ.
Since $\mathbf{w}$ is a vector in $\mathbb{R}^3$, there exist some $\alpha, \beta_1, \beta_2 \in \mathbb{R}$ such that $\mathbf{w} = \alpha\mathbf{u} + \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2$.

ii. We have $\mu\alpha\mathbf{u} + \mu\beta_1\mathbf{v}_1 + \mu\beta_2\mathbf{v}_2 = \mu\mathbf{w} = A\mathbf{w} = \alpha A\mathbf{u} + \beta_1 A\mathbf{v}_1 + \beta_2 A\mathbf{v}_2 = 4\alpha\mathbf{u} + \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2$.
Then $(\mu - 4)\alpha\mathbf{u} + (\mu - 1)\beta_1\mathbf{v}_1 + (\mu - 1)\beta_2\mathbf{v}_2 = \mathbf{0}$.
Since $\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2$ are linearly independent, we have $(\mu - 4)\alpha = (\mu - 1)\beta_2 = (\mu - 1)\beta_3 = 0$.

iii. We verify that $\mu = 4$ or $\mu = 1$:

- Suppose $\mu \neq 4$ and $\mu \neq 1$.
  Then, since $(\mu - 4)\alpha = (\mu - 1)\beta_2 = (\mu - 1)\beta_3 = 0$, we would have $\alpha = \beta = \gamma = 0$.
  Therefore $\mathbf{w} = \alpha\mathbf{u} + \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 = \mathbf{0}$. This is impossible because $\mathbf{w}$ is an eigenvector of A.

iv. Now we have confirmed that $\mu = 4$ or $\mu = 1$.

- (Case 1.) Suppose $\mu = 4$.
  Then since $(\mu - 1)\beta_2 = (\mu - 1)\beta_3 = 0$, we have $\beta = \gamma = 0$.
  Therefore $\mathbf{w} = \alpha\mathbf{u}$.
  Hence $\mathbf{w}$ is a scalar multiple of $\mathbf{u}$.
- (Case 2.) Suppose $\mu = 1$.
  Then since $(\mu - 4)\alpha = 0$, we have $\alpha = 0$.
  Therefore $\mathbf{w} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2$.
  Hence $\mathbf{w}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$.

Hence, in any case $\mathbf{w}$ is a scalar multiple of $\mathbf{u}$ or a linear combination of $\mathbf{v}_1, \mathbf{v}_2$.