1. **Definition. (Nullity, column rank, row rank of a matrix.)**

Let A be a $(p \times q)$-matrix.

(a) *The nullity of A is defined to be the dimension of the null space of A.*
It is denoted by $n(A)$.

(b) *The column rank of A is defined to be the dimension of the column space of A.*
It is denoted by $r_{col}(A)$.

(c) *The row rank of A is defined to be the dimension of the row space of A.*
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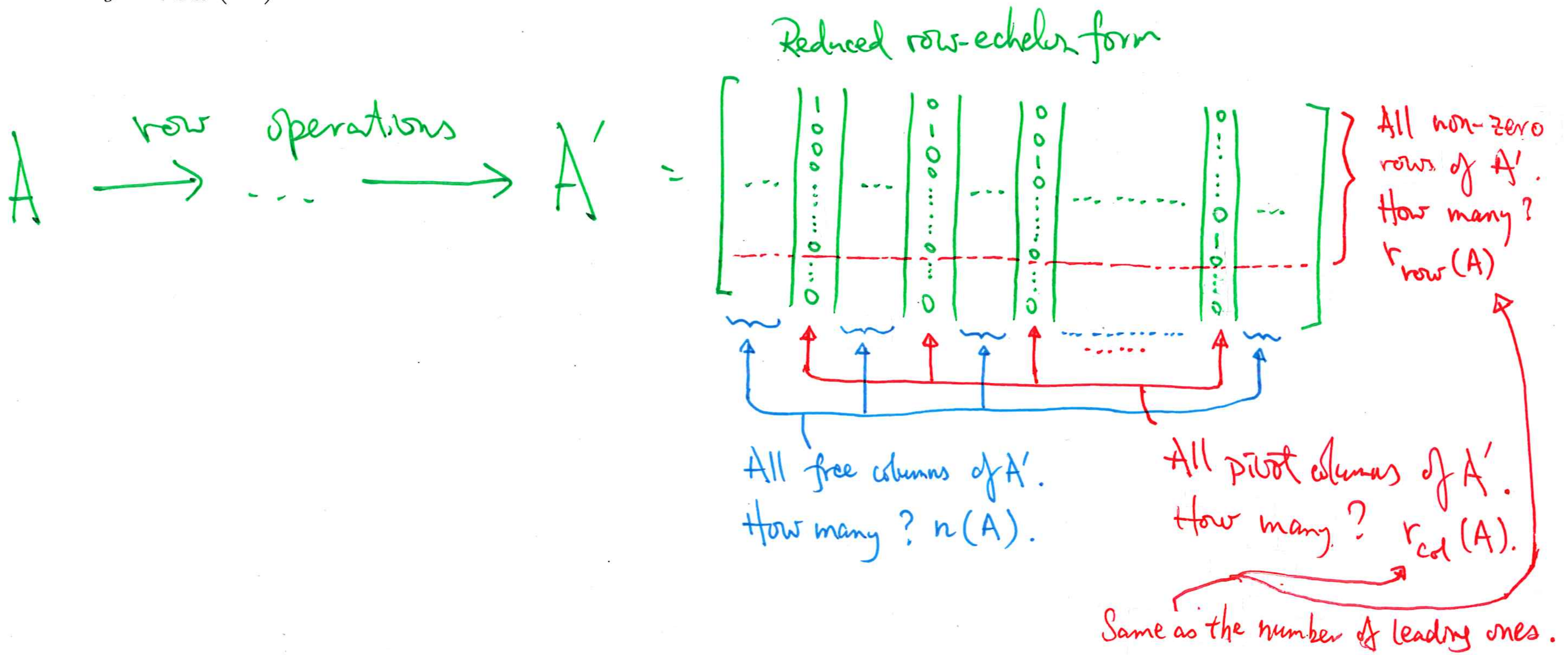

2. **Theorem (K).**

Let A be a $(p \times q)$-matrix.

Suppose A' is the reduced row-echelon form which is row-equivalent to A.

Denote the rank of A' by $r(A)$. (So $r(A)$ is the number of leading ones in A'.)

Then the statements below hold:

(a) $r(A) = r_{col}(A) = r_{row}(A)$.

(b) $n(A) + r(A) = q$.

(c) $r(A^t) = r(A)$, *and* $n(A^t) + r(A) = p$.

Remarks.

- The column space of A is a subspace of $\mathbb{R}^q$ while the row space of A is a subspace of $\mathbb{R}^p$. So despite the equality $r_{col}(A) = r_{row}(A)$, we do not expect these two objects to be 'comparable'.
  In fact, what is important is that despite their distinction as objects, their respective dimensions are the same.

- The equality '$n(A) + r(A) = q$' is referred to as the 'Rank-nullity Formula' (for the matrix A with q columns).

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Denote the rank of A' by $r(A)$. (So $r(A)$ is the number of leading ones in A'.)

Then the statements below hold:

(a) $r(A) = r_{col}(A) = r_{row}(A)$. ← No more than saying these three numbers are the same: Number of leading ones in A', number of pivot columns in A', and number of non-zero rows in A'.

(b) $n(A) + r(A) = q$. ← No more than saying: the number of columns of A is the same as the sum of the number of free columns in A' and the number of pivot columns in A'.

(c) $r(A^t) = r(A)$, and $n(A^t) + r(A) = p$.

Not so immediately apparent.

Immediate consequence of '$r(A^t) = r(A)$'.

Remarks.

- The column space of A is a subspace of $\mathbb{R}^q$ while the row space of A is a subspace of $\mathbb{R}^p$. So despite the equality $r_{col}(A) = r_{row}(A)$, we do not expect these two objects to be 'comparable'.
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- The equality '$n(A) + r(A) = q$' is referred to as the 'Rank-nullity Formula' (for the matrix A with q columns).

3. **Proof of Theorem (K).**

(a) The number of vectors in a basis for $\mathcal{C}(A)$ is the same as the number of pivot columns in A', which is the rank of A'. Hence $r(A) = r_{col}(A)$.

The number of vectors in a basis for $\mathcal{R}(A)$ is the number of non-zero rows in A', which is also the rank of A'. Hence $r(A) = r_{row}(A)$.

(b) The nullity of A is the same as the number of free columns in A'.
Then $n(A) = q - r(A)$.
Therefore $n(A) + r(A) = q$.

(c) Note that $\mathcal{C}(A^t) = \mathcal{R}(A)$.
We have $r(A^t) = r_{col}(A^t) = r_{row}(A) = r(A)$.
Then $n(A^t) + r(A) = n(A^t) + r(A^t) = p$.

4. **Corollary to Theorem (K).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_t$ be vectors in $\mathbb{R}^q$.
Define $U = \left[\ \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{u}_t\ \right]$.
Then the dimension of Span $(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_t\})$ *is $r(U)$.*

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(b) The nullity of A is the same as the number of free columns in A'.
Then $n(A) = q - r(A)$.
Therefore $n(A) + r(A) = q$.

(c) Note that $\mathcal{C}(A^t) = \mathcal{R}(A)$. $\leftarrow$ So $r_{col}(A^t) = \dim(\mathcal{C}(A^t)) = \dim(\mathcal{R}(A)) = r_{row}(A)$.
We have $r(A^t) = r_{col}(A^t) = r_{row}(A) = r(A)$.
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4. **Corollary to Theorem (K).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_t$ *be vectors in* $\mathbb{R}^q$.
Define $U = \left[\, \mathbf{u}_1 \middle| \mathbf{u}_2 \middle| \cdots \middle| \mathbf{u}_t \,\right]$.
Then the dimension of $\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_t\})$ *is* $r(U)$.

5. **Illustrations of the content of Theorem (K).**

(a) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 3 & 4 & 4 & 3 \\ 2 & 2 & 1 & 1 \end{bmatrix}$, and write $B = A^t$.

The reduced row-echelon form A' which is row-equivalent to A is given by $A' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

By direct inspection on A', we see that $r(A) = 3$ and $n(A) = 1$.
As expected from theory, we have $n(A) + r(A) = 4$.

Note that $B = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 1 & 0 & 4 & 2 \\ 1 & -1 & 4 & 1 \\ 1 & 0 & 3 & 1 \end{bmatrix}$.

The reduced row-echelon form B' which is row-equivalent to B is given by $B' = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Note that B' is not the same as the transpose of A'.
However $r(B) = r(B') = 3$; so, as expected from theory, $r(B) = r(A)$.
By direct inspection on B', we see that $r(B) = 3$ and $n(B) = 1$.
As expected from theory, $n(B) + r(B) = 4$.

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By direct inspection on B', we see that $r(B) = 3$ and $n(B) = 1$.
As expected from theory, $n(B) + r(B) = 4$.

(b) Let $A = \begin{bmatrix} 1 & 2 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 & 5 \\ 2 & 6 & 5 & 9 & 6 \end{bmatrix}$, and write $B = A^t$.

The reduced row-echelon form A' which is row-equivalent to A is given by $A' = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 1 & -1 & 4 \end{bmatrix}$.

By direct inspection on A', we see that $r(A) = 3$ and $n(A) = 2$.
As expected from theory, we have $n(A) + r(A) = 5$.

Note that $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 6 \\ 2 & 3 & 5 \\ 3 & 4 & 9 \\ 4 & 5 & 6 \end{bmatrix}$.

The reduced row-echelon form B' which is row-equivalent to B is given by $B' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Note that B' is not the same as the transpose of A'.
However $r(B) = r(B') = 3$; so, as expected from theory, $r(B) = r(A)$.
By direct inspection on B', we see that $r(B) = 3$ and $n(B) = 0$.
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The reduced row-echelon form A' which is row-equivalent to A is given by $A' = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 1 & -1 & 4 \end{bmatrix}$.

By direct inspection on A', we see that $r(A) = 3$ and $n(A) = 2$.
As expected from theory, we have $n(A) + r(A) = 5$.

Note that $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 6 \\ 2 & 3 & 5 \\ 3 & 4 & 9 \\ 4 & 5 & 6 \end{bmatrix}$.

The reduced row-echelon form B' which is row-equivalent to B is given by $B' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Note that B' is not the same as the transpose of A'.
However $r(B) = r(B') = 3$; so, as expected from theory, $r(B) = r(A)$.
By direct inspection on B', we see that $r(B) = 3$ and $n(B) = 0$.
As expected from theory, $n(B) + r(B) = 3$.

(c) Let $A = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{bmatrix}$, and write $B = A^t$.

The reduced row-echelon form A' which is row-equivalent to A is given by $A' = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

By direct inspection on A', we see that $r(A) = 3$ and $n(A) = 4$.
As expected from theory, we have $n(A) + r(A) = 7$.

Note that $B = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & 0 & -4 & -6 \\ -1 & 2 & -1 & -1 \\ 1 & 3 & 3 & 5 \\ 0 & 5 & 2 & 4 \\ 2 & -7 & 1 & 0 \\ 0 & 12 & 5 & 10 \end{bmatrix}$.

The reduced row-echelon form B' which is row-equivalent to B is given by $B' = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Note that B' is not the same as the transpose of A'.
However $r(B) = r(B') = 3$; so, as expected from theory, $r(B) = r(A)$.
By direct inspection on B', we see that $r(B) = 3$ and $n(B) = 1$.
As expected from theory, $n(B) + r(B) = 4$.

(c) Let $A = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{bmatrix}$, and write $B = A^t$.

The reduced row-echelon form A' which is row-equivalent to A is given by $A' = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

By direct inspection on A', we see that $r(A) = 3$ and $n(A) = 4$.
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Note that $B = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & 0 & -4 & -6 \\ -1 & 2 & -1 & -1 \\ 1 & 3 & 3 & 5 \\ 0 & 5 & 2 & 4 \\ 2 & -7 & 1 & 0 \\ 0 & 12 & 5 & 10 \end{bmatrix}$.

The reduced row-echelon form B' which is row-equivalent to B is given by $B' = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Note that B' is not the same as the transpose of A'.
However $r(B) = r(B') = 3$; so, as expected from theory, $r(B) = r(A)$.
By direct inspection on B', we see that $r(B) = 3$ and $n(B) = 1$.
As expected from theory, $n(B) + r(B) = 4$.

6. **Theorem (1).**

Suppose A is a $(p \times q)$-matrix.

Then the inequalities below hold:

(a) $r(A) \leq p$.

(b) $r(A) \leq q$.

(c) $n(A) \geq q - p$.

Proof of Theorem (1).

The first two inequalities follow immediately from the definition of $r(A)$ as the dimension of the column space of A and also as the dimension of the row space of A.

As for the third, it is a consequence of the equality $n(A) = q - r(A)$.

7. **Lemma (2).**

Suppose A is a $(p \times q)$-matrix, and B is an $(q \times s)$-matrix.

Then $\mathcal{N}(B)$ is a subspace of $\mathcal{N}(AB)$.

Proof of Lemma (2).

Suppose A is a $(p \times q)$-matrix, and B is a $(q \times s)$-matrix.

By definition, AB is an $(p \times s)$-matrix.

Note that $\mathcal{N}(B), \mathcal{N}(AB)$ are both subspaces of $\mathbb{R}^s$.

[We verify that *for any* $\mathbf{v} \in \mathbb{R}^s$, *if* $\mathbf{v} \in \mathcal{N}(B)$ *then* $\mathbf{v} \in \mathcal{N}(AB)$.]

Pick any vector $\mathbf{v} \in \mathbb{R}^s$. Suppose $\mathbf{v} \in \mathcal{N}(B)$. Then by definition, $B\mathbf{v} = \mathbf{0}_q$.

We have $(AB)\mathbf{v} = A(B\mathbf{v}) = A\mathbf{0}_q = \mathbf{0}_p$. Then by definition $\mathbf{v} \in \mathcal{N}(AB)$.

It follows that $\mathcal{N}(B)$ is a subspace of $\mathcal{N}(AB)$.

8. **Theorem (3).**

Suppose A is a $(p \times q)$-matrix, and B is an $(q \times s)$- matrix.

Then the inequalities below hold:

(a) $n(B) \leq n(AB)$.

(b) $r(AB) \leq r(B)$.

(c) $r(AB) \leq r(A)$.

(d) $n(A) + s \leq n(AB) + q$.

9. **Proof of Theorem (3).**

Suppose A is a $(p \times q)$-matrix, and B is a $(q \times s)$-matrix.

(a) By Lemma (2), $\mathcal{N}(B)$ is a subspace of $\mathcal{N}(AB)$.

Then

$$n(B) = \dim(\mathcal{N}(B)) \leq \dim(\mathcal{N}(AB)) = n(AB).$$

8. **Theorem (3).**

Suppose A is a $(p \times q)$-matrix, and B is an $(q \times s)$- matrix.

Then the inequalities below hold:

(a) $n(B) \leq n(AB)$.

(b) $r(AB) \leq r(B)$.

So $r(AB)$ is of value at most the minimum of $r(A), r(B)$.

(c) $r(AB) \leq r(A)$.

(d) $n(A) + s \leq n(AB) + q$.

9. **Proof of Theorem (3).**

Suppose A is a $(p \times q)$-matrix, and B is a $(q \times s)$-matrix.

(a) By Lemma (2), $\mathcal{N}(B)$ is a subspace of $\mathcal{N}(AB)$.

Then

$$n(B) = \dim(\mathcal{N}(B)) \leq \dim(\mathcal{N}(AB)) = n(AB).$$

(b) By the Rank-nullity Formula, we have

$$n(B) + r(B) = s, \quad \text{and} \quad n(AB) + r(AB) = s.$$

Then

$$r(AB) = s - n(AB) \leq s - n(B) = r(B).$$

(c) Note that $B^t A^t = (AB)^t$. Then, also by Lemma (2), $\mathcal{N}(A^t)$ is a subspace of $\mathcal{N}((AB)^t)$.

Therefore

$$n(A^t) = \dim(\mathcal{N}\big(A^t\big)) \leq \dim(\mathcal{N}\big((AB)^t\big)) = n((AB)^t).$$

By the Rank-nullity Formula, we have

$$n(A^t) + r(A^t) = p \quad \text{and} \quad n((AB)^t) + r((AB)^t) = p.$$

Then

$$r(AB) = r((AB)^t) = p - n((AB)^t) \leq p - n(A^t) = r(A^t) = r(A).$$

(d) Again by the Rank-nullity Formula, we have

$$n(A) + r(A) = n \quad \text{and} \quad n(AB) + r(AB) = s.$$

Then

$$s - n(AB) = r(AB) \leq r(A) = q - n(A).$$

Therefore

$$n(A) + s \leq n(AB) + q.$$