1. Recall the definition for the notion of *basis for a subspace of* $\mathbb{R}^n$.

   Let V be a subspace of $\mathbb{R}^n$.
   We declare that if V is the zero subspace of $\mathbb{R}^n$ then the empty set is the basis for V.
   From now on suppose V is not the zero subspace of $\mathbb{R}^n$.
   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are vectors in V.
   The vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are said to constitute a basis for V (or the set $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\}$ is said to be a basis for V) if and only if both of the statements (BL), (BS) *below hold:*

   (BL) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ *are linearly independent.*

   (BS) *Every vector in V is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$.*

   Also recall Theorem (B) below, from the handout *Bases for subspaces of* $\mathbb{R}^n$:

   Any two bases for a subspace of $\mathbb{R}^n$ have the same number of vectors.

   Further recall Theorem (C) below, from the handout *Bases for subspaces of* $\mathbb{R}^n$:

   Suppose V is a non-zero subspace of $\mathbb{R}^n$. Then V has a basis which consists of at least one and at most n vectors in $\mathbb{R}^n$.

   They combine to make sense of the definition for the notion of *dimension*, introduced below.

2. **Definition. (Dimension.)**

   Let V be a subspace of $\mathbb{R}^n$.

   When V is not the zero subspace of $\mathbb{R}^n$, the number of vectors in a basis for V is called the dimension of V. When this number is p, we write $\dim(V) = p$, and we refer to V as a p-dimensional subspace of $\mathbb{R}^n$.

   We declare the dimension of the zero subspace of $\mathbb{R}^n$ to be 0.

   Remark. By definition, when V is a subspace of $\mathbb{R}^n$, it happens that $\dim(V) \leq n$. (Why? A basis for V is necessarily a collection of linearly independent vectors in $\mathbb{R}^n$; there are at most n vectors in such a collection.)

3. **Theorem (1).**

   $\mathbb{R}^n$ is an n-dimensional subspace of $\mathbb{R}^n$.

   Proof of Theorem (1).

   The n vectors $\mathbf{e}_1^{(n)}, \mathbf{e}_2^{(n)}, \cdots, \mathbf{e}_n^{(n)}$ constitute a basis for $\mathbb{R}^n$.

4. **Examples.**

   (a) Let $A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix}$, and $V = \mathcal{N}(A)$.

   After some work, we find that a basis for V is constituted by the vector $\mathbf{u}$, in which $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ -4 \\ 1 \end{bmatrix}$.

   Then $\dim(V) = 1$.

   (b) Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix}$, and $V = \mathcal{N}(A)$.

   After some work, we find that a basis for V is constituted by the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, in which $\mathbf{u}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$,

   $\mathbf{u}_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

   Hence $\dim(V) = 3$.

   (c) Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$, and $V = \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\})$.

   After some work, we find that a basis for V is constituted by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.
   Hence $\dim(V) = 3$.

(d) Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 7 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 2 \\ 5 \\ -1 \\ 9 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ -1 \\ -5 \\ 2 \\ 0 \end{bmatrix}$ and $V = \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\})$.

After some work, we find that a basis for V is constituted by $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$, in which $\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{t}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$.

Hence $\dim(V) = 3$.

5. Recall the Replacement Theorem (Theorem (F)) from the handout *More on minimal spanning set*:

   Let W be a subspace of $\mathbb{R}^n$.
   Let $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ be vectors in W. Suppose none of these vectors is the zero vector.
   Suppose $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$ are linearly independent.
   Further suppose $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ constitute a basis for W.
   Then, $q \geq p$, and there is a basis for W which is constituted by $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$ together with some $q-p$ vectors from amongst $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$.

   A consequence of this result is Theorem (2).

6. **Theorem (2).**

   $\mathbb{R}^n$ is the only n-dimensional subspace of $\mathbb{R}^n$.

   Proof of Theorem (2).

   [We are going to prove the statement 'if V is an n-dimensional subspace of $\mathbb{R}^n$ then $V = \mathbb{R}^n$.']

   Let V be a subspace of $\mathbb{R}^n$. Suppose that $\dim(V) = n$.

   There is some basis with n vectors in V, say, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$. They are n linearly independent vectors in $\mathbb{R}^n$.

   Note that $\mathbf{e}_1^{(n)}, \mathbf{e}_2^{(n)}, \cdots, \mathbf{e}_n^{(n)}$ constitute some basis for $\mathbb{R}^n$.

   Then by the Replacement Theorem, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$, together with possibly some vectors from amongst $\mathbf{e}_1^{(n)}, \mathbf{e}_2^{(n)}, \cdots, \mathbf{e}_n^{(n)}$, constitute a basis for $\mathbb{R}^n$.
   However, since $\dim(\mathbb{R}^n) = n$, the n vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ already constitute a basis for $\mathbb{R}^n$.

   It follows that $\mathbb{R}^n = \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}) = V$.

   Remark. This argument can be generalized to give an important theoretical tool with wide applications. See the handout *Inequalities on dimension*.

7. Recall Theorem (G) from the handout *More on minimal spanning set*:

   Let W be a non-zero subspace of $\mathbb{R}^n$, and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ be vectors in W.
   Further suppose that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent.
   Then, there is some basis for W, which is constituted of at most n vectors, amongst them being the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$.

   According to this result, whenever we have k linearly independent vectors in a subspace, say, W, of $\mathbb{R}^n$, these k vectors will be part of a basis for W. Then it is necessary for W to have dimension at least k.

   Out of this discussion, we obtain Theorem (H) below.

8. **Theorem (H).**

   Let W be a p-dimensional subspace of $\mathbb{R}^n$, and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ be vectors in W.

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are linearly independent. Then $k \leq p$.

   Proof of Theorem (H).

   By assumption there is some basis for W which is constituted by p vectors in W, amongst them being these k vectors. Then $k \leq p$.

   Remark. This is a generalization of the result below, from the handout *Linear dependence and linear independence*:

   Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_\ell$ be vectors in $\mathbb{R}^m$. Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_\ell$ are linearly independent. Then $\ell \leq m$.

9. According to logic, Theorem (H) is saying the same thing as Corollary to Theorem (H) below:

   Corollary to Theorem (H).

   Any $p+1$ or more vectors in a p-dimensional subspace of $\mathbb{R}^n$ are linearly dependent.