MATH1030 Further examples on construction of bases through row spaces.

1. Consider each of the collection of vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots$ below. Write $V = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots\})$.

   Find a basis for V by applying row operations on the matrix $\begin{bmatrix} \mathbf{u}_1{}^t \\ \hline \mathbf{u}_2{}^t \\ \hline \mathbf{u}_3{}^t \\ \hline \vdots \end{bmatrix}$

   (a) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 4 \\ -3 \\ 4 \\ 5 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 5 \\ 10 \\ -8 \\ 11 \\ 12 \end{bmatrix}$.

   (b) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 7 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 1 \\ 5 \\ -7 \\ 1 \end{bmatrix}$.

   (c) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ -7 \\ 6 \\ -11 \\ -2 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$.

   (d) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 7 \\ 1 \\ -1 \\ 4 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$.

   (e) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 4 \\ 8 \\ 0 \\ -4 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 0 \\ 9 \\ -4 \\ 8 \end{bmatrix}$, $\mathbf{u}_6 = \begin{bmatrix} 7 \\ -13 \\ 12 \\ -31 \end{bmatrix}$, $\mathbf{u}_7 = \begin{bmatrix} -9 \\ 7 \\ -8 \\ 37 \end{bmatrix}$.

Answer.

1. (a) A basis for V is constituted by $\mathbf{t}_1, \mathbf{t}_2$, in which $\mathbf{t}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 4 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$.

Reason:

$$\left[\begin{array}{c} \mathbf{u}_1{}^t \\ \hline \mathbf{u}_2{}^t \\ \hline \mathbf{u}_3{}^t \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 2 & 0 & -1 & 4 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) A basis for V is constituted by $\mathbf{t}_1, \mathbf{t}_2$, in which $\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \\ 1 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \\ 4 \end{bmatrix}$.

Reason:

$$\left[\begin{array}{c} \mathbf{u}_1{}^t \\ \hline \mathbf{u}_2{}^t \\ \hline \mathbf{u}_3{}^t \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -3 & 1 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) A basis for V is constituted by $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$, in which $\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1/17 \\ 30/17 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 25/17 \\ -2/17 \end{bmatrix}$, $\mathbf{t}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2/17 \\ -8/17 \end{bmatrix}$.

Reason:

$$\left[\begin{array}{c} \mathbf{u}_1{}^t \\ \hline \mathbf{u}_2{}^t \\ \hline \mathbf{u}_3{}^t \\ \hline \mathbf{u}_4{}^t \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1/17 & 30/17 \\ 0 & 1 & 0 & 25/17 & -2/17 \\ 0 & 0 & 1 & -2/17 & -8/17 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(d) A basis for V is constituted by $\mathbf{t}_1, \mathbf{t}_2$, in which $\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ -1/2 \\ 1/2 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 0 \\ 1 \\ 5/2 \\ 1/2 \end{bmatrix}$.

Reason:

$$\left[\begin{array}{c} \mathbf{u}_1{}^t \\ \hline \mathbf{u}_2{}^t \\ \hline \mathbf{u}_3{}^t \\ \hline \mathbf{u}_4{}^t \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 5/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(e) A basis for V is constituted by $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$, in which $\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -31/7 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 12/7 \end{bmatrix}$, $\mathbf{t}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 13/7 \end{bmatrix}$.

Reason:

$$\left[\begin{array}{c} \mathbf{u}_1{}^t \\ \hline \mathbf{u}_2{}^t \\ \hline \mathbf{u}_3{}^t \\ \hline \mathbf{u}_4{}^t \\ \hline \mathbf{u}_5{}^t \\ \hline \mathbf{u}_6{}^t \\ \hline \mathbf{u}_7{}^t \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -31/7 \\ 0 & 1 & 0 & 12/7 \\ 0 & 0 & 1 & 13/7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$