1. Recall Theorem (E) from the handout *Minimal spanning set*:

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$ *be vectors in* $\mathbb{R}^n$, *and* U *be the* $(n \times q)$*-matrix given by* $U = \left[\ \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{u}_q\ \right]$.
Let $V = \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q\})$.

Denote by U' *the reduced row-echelon form which is row-equivalent to* U.
Denote the j*-th column of* U' *by* $\mathbf{u}'_j$.
Denote the rank of U' *by* r. *Suppose* $r \geq 1$, *and label the pivot columns of* U', *from left to right, by* $d_1, d_2, \cdots, d_r$.

Then $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$ *constitute a basis for* V.

Moreover, for each $j = 1, 2, \cdots, q$,

the vector $\mathbf{u}_j$ *is the linear combination of* $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$ *and the respective scalars* $\alpha_1, \alpha_2, \cdots, \alpha_r$

if and only if

$$\mathbf{u}'_j = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We give some applications of Theorem (E) in the theory for the notion of *basis*.

2. **Lemma (1).**
Let V, W be subspaces of $\mathbb{R}^n$.

Define the set $V + W$ by

$$V + W = \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{array}{l} \textit{There exist some } y \in V,\ z \in W \\ \textit{such that } x = y + z \end{array} \right\}.$$

Then $V + W$ is a subspace of $\mathbb{R}^n$.

Remark.
$V + W$ is called the sum of V and W.

Proof of Lemma (1). Exercise.

3. An immediate application of Theorem (E) in helping us determine a basis for the sum of two subspaces of $\mathbb{R}^n$ when a basis for each subspace concerned is already known (or, more generally, when each subspace concerned has already been expressed as the span of several vectors in $\mathbb{R}^n$).

2. **Lemma (1).**

Let V, W be subspaces of $\mathbb{R}^n$.

Define the set $V + W$ by

$$V + W = \left\{ \mathbf{x} \in \mathbb{R}^n : \boxed{\begin{array}{l} \text{There exist some } y \in V,\ z \in W \\ \text{such that } x = y + z \end{array}} \right\}.$$

In plain words, this 'selection criterion' reads: 'x is a vector of the form $y+z$ in which $y \in V$ and $z \in W$.'

This turns out to be the same as: 'x is a linear combination of vectors belonging to V or W.'

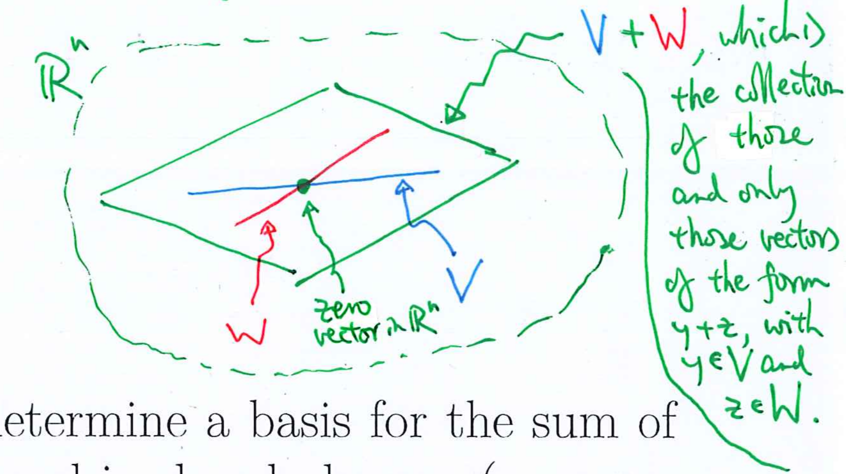
Then $V + W$ is a subspace of $\mathbb{R}^n$.

Remark.

$V + W$ is called the sum of V and W.

Proof of Lemma (1). Exercise.

Illustration of the idea in this definition



3. An immediate application of Theorem (E) in helping us determine a basis for the sum of two subspaces of $\mathbb{R}^n$ when a basis for each subspace concerned is already known (or, more generally, when each subspace concerned has already been expressed as the span of several vectors in $\mathbb{R}^n$).

Theorem (2).

Let V, W be subspaces of $\mathbb{R}^n$. Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p \in V$, $\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots u_{p+q} \in W$.
Suppose none of these vectors is the zero vector.

Suppose $V = \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\})$ and $W = \mathsf{Span}\,(\{\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}\})$.

Then the statements below hold:

(a) $V + W = \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p, \mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}\})$.

(b) *Suppose* $U = \left[\ \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{u}_p \,\middle|\, \mathbf{u}_{p+1} \,\middle|\, \mathbf{u}_{p+2} \,\middle|\, \cdots \,\middle|\, \mathbf{u}_{p+q}\ \right]$.
Denote by U' the reduced row-echelon form which is row-equivalent to U.
Denote the rank of U' by r.
Label the pivot columns of U' from left to right by $d_1, d_2, \cdots, d_r$.

Then a basis for $V + W$ is constituted by $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$.

(c) *Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for V, and $\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}$ constitute a basis for W.*
Then $d_j = j$ for each $j = 1, 2, \cdots, p$. Moreover, $r \geq p$.

Further write $s = r - p$, and $k_1 = d_{p+1} - p$, $k_2 = d_{p+2} - p$, ..., $k_s = d_r - p$.
Then a basis for $V + W$ is constituted by $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p, \mathbf{u}_{p+k_1}, \mathbf{u}_{p+k_2}, \cdots, \mathbf{u}_{p+k_s}$.

Proof of Theorem (2). Exercise. (Apply Lemma (1) and Theorem (E). The hard work has been done in the proof of Theorem (E).)

Theorem (2).

Let V, W be subspaces of $\mathbb{R}^n$. Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p \in V$, $\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots u_{p+q} \in W$. Suppose none of these vectors is the zero vector.

Suppose $V = \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\})$ and $W = \mathsf{Span}\,(\{\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}\})$.

Then the statements below hold:

(a) $V + W = \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p, \mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}\})$.

V+W is just the collection of all linear combinations that can be formed with vectors belonging to V and vectors belonging to W.

This is an immediate consequence of Theorem (E).

(b) *Suppose* $U = [\, \mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\cdots|\, \mathbf{u}_p \,|\, \mathbf{u}_{p+1} \,|\, \mathbf{u}_{p+2} \,|\cdots|\, \mathbf{u}_{p+q} \,]$.
Denote by U' the reduced row-echelon form which is row-equivalent to U.
Denote the rank of U' by r.
Label the pivot columns of U' from left to right by $d_1, d_2, \cdots, d_r$.

Then a basis for $V + W$ is constituted by $\mathbf{u}_{d_1}, \mathbf{u}_{d_2}, \cdots, \mathbf{u}_{d_r}$.

Beware: although $u_1, u_2, \ldots, u_p$ are linearly independent, and $u_{p+1}, u_{p+2}, \ldots, u_{p+q}$ are linearly independent, there is no guarantee that $u_1, \ldots, u_p, u_{p+1}, \ldots, u_{p+q}$ are linearly independent.

(c) *Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for V, and $\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}$ constitute a basis for W.*
Then $d_j = j$ for each $j = 1, 2, \cdots, p$. Moreover, $r \geq p$.
Further write $s = r - p$, and $k_1 = d_{p+1} - p$, $k_2 = d_{p+2} - p$, ..., $k_s = d_r - p$.
Then a basis for $V + W$ is constituted by $\boxed{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p}, \boxed{\mathbf{u}_{p+k_1}}, \boxed{\mathbf{u}_{p+k_2}}, \cdots, \boxed{\mathbf{u}_{p+k_s}}$.

Due to linear independence of $u_1, u_2, \ldots, u_p$.

Nonetheless, $u_1, u_2, \ldots, u_p$, supplemented by possibly some amongst $u_{p+1}, u_{p+2}, \ldots, u_{p+q}$, give a basis for V+W.

Proof of Theorem (2). Exercise. (Apply Lemma (1) and Theorem (E). The hard work has been done in the proof of Theorem (E).)

4. **Illustrations for Theorem (2).**

(a) *Let*

$$\mathbf{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{s}_3 = \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix}, \mathbf{t}_1 = \begin{bmatrix} -1 \\ 1 \\ 5 \\ 5 \end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{t}_3 = \begin{bmatrix} -1 \\ 0 \\ 9 \\ 0 \end{bmatrix}, \mathbf{t}_4 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

Define $V = \mathsf{Span}\,(\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\})$, $W = \mathsf{Span}\,(\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4\})$.

We want to find a basis for $V + W$.

Define $U = \left[\, \mathbf{s}_1 \,\middle|\, \mathbf{s}_2 \,\middle|\, \mathbf{s}_3 \,\middle|\, \mathbf{t}_1 \,\middle|\, \mathbf{t}_2 \,\middle|\, \mathbf{t}_3 \,\middle|\, \mathbf{t}_4 \,\right]$.

We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \left[\begin{array}{ccc|cccc} 1 & 2 & 7 & -1 & 1 & -1 & 3 \\ 1 & 1 & 3 & 1 & 1 & 0 & 2 \\ 3 & 2 & 5 & 5 & -1 & 9 & 1 \\ 1 & -1 & -5 & 5 & 2 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow U' = \left[\begin{array}{ccc|cccc} 1 & 0 & -1 & 3 & 0 & 3 & 0 \\ 0 & 1 & 4 & -2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The rank of U' is 3, and the pivot columns are the first, second, and fifth columns.

Hence a basis for $V + W$ is constituted by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{t}_2$.

4. **Illustrations for Theorem (2).**

(a) *Let*

$$\mathbf{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{s}_3 = \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix}, \mathbf{t}_1 = \begin{bmatrix} -1 \\ 1 \\ 5 \\ 5 \end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{t}_3 = \begin{bmatrix} -1 \\ 0 \\ 9 \\ 0 \end{bmatrix}, \mathbf{t}_4 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

Define $V = \mathsf{Span}\,(\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\})$, $W = \mathsf{Span}\,(\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4\})$.

We want to find a basis for $V + W$.

Define $U = \left[\, \underset{u_1}{\mathbf{s}_1} \,\middle|\, \underset{u_2}{\mathbf{s}_2} \,\middle|\, \underset{u_3}{\mathbf{s}_3} \,\middle|\, \underset{u_4}{\mathbf{t}_1} \,\middle|\, \underset{u_5}{\mathbf{t}_2} \,\middle|\, \underset{u_6}{\mathbf{t}_3} \,\middle|\, \underset{u_7}{\mathbf{t}_4} \,\right]$.

We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{array}{c} \left[\begin{array}{c|c|c|c|c|c|c} 1 & 2 & 7 & -1 & 1 & -1 & 3 \\ 1 & 1 & 3 & 1 & 1 & 0 & 2 \\ 3 & 2 & 5 & 5 & -1 & 9 & 1 \\ 1 & -1 & -5 & 5 & 2 & 0 & 1 \end{array}\right] \\ \begin{array}{ccccccc} u_{d_1} & u_{d_2} & & & u_{d_3} & & \end{array} \end{array} \longrightarrow \cdots\cdots \longrightarrow U' = \begin{array}{c} \left[\begin{array}{c|c|c|c|c|c|c} 1 & 0 & -1 & 3 & 0 & 3 & 0 \\ 0 & 1 & 4 & -2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccccc} u'_{d_1} & u'_{d_2} & & & u'_{d_3} & & \end{array} \end{array}$$

The rank of U' is 3, and the pivot columns are the first, second, and fifth columns.

Hence a basis for $V + W$ is constituted by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{t}_2$.

(b) *Let*

$$\mathbf{s}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \mathbf{s}_3 = \begin{bmatrix} -3 \\ -4 \\ 0 \\ -6 \\ 1 \end{bmatrix}, \mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \mathbf{t}_3 = \begin{bmatrix} 3 \\ 8 \\ 2 \\ 5 \\ -1 \end{bmatrix}.$$

Define $V = \mathsf{Span}\,(\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\})$, $W = \mathsf{Span}\,(\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\})$.

We want to find a basis for $V + W$.

Define $U = \left[\, \mathbf{s}_1 \middle| \mathbf{s}_2 \middle| \mathbf{s}_3 \middle| \mathbf{t}_1 \middle| \mathbf{t}_2 \middle| \mathbf{t}_3 \,\right]$.
We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \left[\begin{array}{ccc|ccc} -1 & 1 & -3 & 1 & 1 & 3 \\ 1 & -2 & -4 & 0 & 1 & 8 \\ 1 & 0 & 0 & 0 & 0 & 2 \\ -2 & 3 & -6 & 2 & 2 & 5 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow U' = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The rank of U' is 5, and the pivot columns are the first five columns.

Hence $V + W = \mathbb{R}^5$, and a basis for $V + W$ is constituted by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{t}_1, \mathbf{t}_2$.

(b) *Let*

$$\mathbf{s}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \mathbf{s}_3 = \begin{bmatrix} -3 \\ -4 \\ 0 \\ -6 \\ 1 \end{bmatrix}, \mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \mathbf{t}_3 = \begin{bmatrix} 3 \\ 8 \\ 2 \\ 5 \\ -1 \end{bmatrix}.$$

Define $V = \mathsf{Span}\,(\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\})$, $W = \mathsf{Span}\,(\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\})$.

We want to find a basis for $V + W$.

Define $U = \left[\, \overset{u_1}{\mathbf{s}_1} \,\middle|\, \overset{u_2}{\mathbf{s}_2} \,\middle|\, \overset{u_3}{\mathbf{s}_3} \,\middle|\, \overset{u_4}{\mathbf{t}_1} \,\middle|\, \overset{u_5}{\mathbf{t}_2} \,\middle|\, \overset{u_6}{\mathbf{t}_3} \,\right]$.

We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{array}{c} \left[\begin{array}{c|c|c|c|c|c} -1 & 1 & -3 & 1 & 1 & 3 \\ 1 & -2 & -4 & 0 & 1 & 8 \\ 1 & 0 & 0 & 0 & 0 & 2 \\ -2 & 3 & -6 & 2 & 2 & 5 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \\ \begin{array}{cccccc} u_{d_1} & u_{d_2} & u_{d_3} & u_{d_4} & u_{d_5} & \end{array} \end{array} \longrightarrow \cdots\cdots \longrightarrow U' = \begin{array}{c} \left[\begin{array}{c|c|c|c|c|c} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \\ \begin{array}{cccccc} u'_{d_1} & u'_{d_2} & u'_{d_3} & u'_{d_4} & u'_{d_5} & \end{array} \end{array}$$

The rank of U' is 5, and the pivot columns are the first five columns.

Hence $V + W = \mathbb{R}^5$, and a basis for $V + W$ is constituted by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{t}_1, \mathbf{t}_2$.

(c) *Let*

$$\mathbf{s}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -3 \\ 3 \end{bmatrix}, \mathbf{s}_3 = \begin{bmatrix} 7 \\ -12 \\ 5 \\ -12 \\ 9 \end{bmatrix}, \mathbf{s}_4 = \begin{bmatrix} -3 \\ 4 \\ -1 \\ 4 \\ -5 \end{bmatrix},$$

and

$$\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix} 2 \\ -4 \\ 1 \\ -5 \\ 6 \end{bmatrix}, \mathbf{t}_3 = \begin{bmatrix} 4 \\ -7 \\ 3 \\ -7 \\ 5 \end{bmatrix}, \mathbf{t}_4 = \begin{bmatrix} 2 \\ -5 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \mathbf{t}_5 = \begin{bmatrix} 6 \\ -9 \\ 4 \\ -10 \\ 7 \end{bmatrix}, \mathbf{t}_6 = \begin{bmatrix} 4 \\ -7 \\ 3 \\ -6 \\ 5 \end{bmatrix}.$$

Define

$$V = \mathsf{Span}\left(\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4\}\right), \quad W = \mathsf{Span}\left(\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5, \mathbf{t}_6\}\right).$$

We want to find a basis for $V + W$.

Define

$$U = \left[\, \mathbf{s}_1 \,\middle|\, \mathbf{s}_2 \,\middle|\, \mathbf{s}_3 \,\middle|\, \mathbf{s}_4 \,\middle|\, \mathbf{t}_1 \,\middle|\, \mathbf{t}_2 \,\middle|\, \mathbf{t}_3 \,\middle|\, \mathbf{t}_4 \,\middle|\, \mathbf{t}_5 \,\middle|\, \mathbf{t}_6 \,\right].$$

(c) *Let*

$$\mathbf{s}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -3 \\ 3 \end{bmatrix}, \mathbf{s}_3 = \begin{bmatrix} 7 \\ -12 \\ 5 \\ -12 \\ 9 \end{bmatrix}, \mathbf{s}_4 = \begin{bmatrix} -3 \\ 4 \\ -1 \\ 4 \\ -5 \end{bmatrix},$$

and

$$\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix} 2 \\ -4 \\ 1 \\ -5 \\ 6 \end{bmatrix}, \mathbf{t}_3 = \begin{bmatrix} 4 \\ -7 \\ 3 \\ -7 \\ 5 \end{bmatrix}, \mathbf{t}_4 = \begin{bmatrix} 2 \\ -5 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \mathbf{t}_5 = \begin{bmatrix} 6 \\ -9 \\ 4 \\ -10 \\ 7 \end{bmatrix}, \mathbf{t}_6 = \begin{bmatrix} 4 \\ -7 \\ 3 \\ -6 \\ 5 \end{bmatrix}.$$

Define

$$V = \mathsf{Span}\left(\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4\}\right), \quad W = \mathsf{Span}\left(\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5, \mathbf{t}_6\}\right).$$

We want to find a basis for $V + W$.

Define

$$U = \left[\, \underset{u_1}{\mathbf{s}_1} \,\middle|\, \underset{u_2}{\mathbf{s}_2} \,\middle|\, \underset{u_3}{\mathbf{s}_3} \,\middle|\, \underset{u_4}{\mathbf{s}_4} \,\middle|\, \underset{u_5}{\mathbf{t}_1} \,\middle|\, \underset{u_6}{\mathbf{t}_2} \,\middle|\, \underset{u_7}{\mathbf{t}_3} \,\middle|\, \underset{u_8}{\mathbf{t}_4} \,\middle|\, \underset{u_9}{\mathbf{t}_5} \,\middle|\, \underset{u_{10}}{\mathbf{t}_6} \,\right].$$

We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \left[\begin{array}{cccc|cccccc} 1 & 2 & 7 & -3 & 1 & 2 & 4 & 2 & 6 & 4 \\ -2 & -3 & -12 & 4 & 0 & -4 & -7 & -5 & -9 & -7 \\ 1 & 1 & 5 & -1 & 0 & 1 & 3 & 2 & 4 & 3 \\ -2 & -3 & -12 & 4 & -1 & -5 & -7 & -3 & -10 & -6 \\ 1 & 3 & 9 & -5 & 1 & 6 & 5 & 3 & 7 & 5 \end{array}\right]$$

$$\longrightarrow \cdots\cdots \longrightarrow U' = \left[\begin{array}{cccc|cccccc} 1 & 0 & 3 & 1 & 0 & -1 & 2 & 0 & 3 & 1 \\ 0 & 1 & 2 & -2 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The rank of U' is 4, and the pivot columns are the first, second, fifth, eighth columns.

Hence a basis for $V + W$ is constituted by

$$\mathbf{s}_1, \mathbf{s}_2, \mathbf{t}_1, \mathbf{t}_4.$$

Remark.

Suppose V, W are respectively given as the null spaces of some matrices with n columns. Then we first obtain a basis for V and a basis for W, and then apply Theorem (2) to obtain a basis for $V + W$.

We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{array}{c} \begin{array}{cccccccccc} u_{d_1} & u_{d_2} & & & u_{d_3} & & & u_{d_4} & & \end{array} \\ \left[\begin{array}{cccc|cccccc} 1 & 2 & 7 & -3 & 1 & 2 & 4 & 2 & 6 & 4 \\ -2 & -3 & -12 & 4 & 0 & -4 & -7 & -5 & -9 & -7 \\ 1 & 1 & 5 & -1 & 0 & 1 & 3 & 2 & 4 & 3 \\ -2 & -3 & -12 & 4 & -1 & -5 & -7 & -3 & -10 & -6 \\ 1 & 3 & 9 & -5 & 1 & 6 & 5 & 3 & 7 & 5 \end{array}\right] \end{array}$$

$$\longrightarrow \cdots\cdots \longrightarrow U' = \begin{array}{c} \left[\begin{array}{cccc|cccccc} 1 & 0 & 3 & 1 & 0 & -1 & 2 & 0 & 3 & 1 \\ 0 & 1 & 2 & -2 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccccccc} u'_{d_1} & u'_{d_2} & & & u'_{d_3} & & & u'_{d_4} & & \end{array} \end{array}$$

The rank of U' is 4, and the pivot columns are the first, second, fifth, eighth columns.

Hence a basis for $V + W$ is constituted by

$$\mathbf{s}_1, \mathbf{s}_2, \mathbf{t}_1, \mathbf{t}_4.$$

Remark.

Suppose V, W are respectively given as the null spaces of some matrices with n columns. Then we first obtain a basis for V and a basis for W, and then apply Theorem (2) to obtain a basis for $V + W$.

5. **Theorem (F). (Replacement Theorem in the context of subspaces of $\mathbb{R}^n$.)**

Let W be a subspace of $\mathbb{R}^n$.

Let $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ be vectors in W.

Suppose none of these vectors is the zero vector.

Suppose $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$ are linearly independent.

Further suppose $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ constitute a basis for W.

Then:

- $q \geq p$, *and*
- *there is a basis for W which is constituted by:*
  - $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$ *together with*
  - *some $q - p$ vectors from amongst $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$.*

Remark.

In plain words, the conclusion in this result says that

> *the linearly independent vectors $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$ in W (which do not necessarily constitute a basis for W because there may be not enough of them to 'span' every vector in W) can be used for 'replacing' p vectors from amongst any given basis for W, say, $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$.*

6. **Proof of Theorem (F).**

Let W be a subspace of $\mathbb{R}^n$.

Let $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ be vectors in W.

Suppose none of these vectors is the zero vector.

Suppose $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$ are linearly independent.

Further suppose $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ constitute a basis for W.

Write $\mathbf{u}_k = \mathbf{s}_k$ for each $k = 1, 2, \cdots, p$, and write $\mathbf{u}_{p+\ell} = \mathbf{t}_\ell$ for each $\ell = 1, 2, \cdots, q$.

Define

$$V = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\}).$$

By assumption,

$$W = \mathsf{Span}\ (\{\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}\}).$$

We have

$$V + W = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p, \mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}\}).$$

Then, by assumption,

$$V + W = \mathsf{Span}\ (\{\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}\}) = W.$$

The conclusion then follows from an application of Theorem (2).

6. **Proof of Theorem (F).**

Let W be a subspace of $\mathbb{R}^n$.

Let $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ be vectors in W.

Suppose none of these vectors is the zero vector.

Suppose $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$ are linearly independent.

Further suppose $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ constitute a basis for W.

Write $\mathbf{u}_k = \mathbf{s}_k$ for each $k = 1, 2, \cdots, p$, and write $\mathbf{u}_{p+\ell} = \mathbf{t}_\ell$ for each $\ell = 1, 2, \cdots, q$.

Define

$$V = \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\}).$$

(#) $u_1, u_2, \ldots, u_p$ is a basis for V.

By assumption,

$$W = \mathsf{Span}\,(\{\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}\}).$$

(##) $u_{p+1}, u_{p+2}, \ldots, u_{p+q}$ is of course a basis for W.

We have

$$V + W = \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p, \mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}\}).$$

Then, by assumption,

$$V + W = \mathsf{Span}\,(\{\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}\}) = W.$$

The conclusion then follows from an application of Theorem (2).

Theorem (2) tells us that because of (#), (##), a basis for $V+W$ is given by $u_1, u_2, \ldots, u_p$ and supplemented by exactly some $q-p$ vectors from $u_{p+1}, u_{p+2}, \ldots, u_{p+q}$. But $V+W$ is also W. Done.

7. **Illustrations for Theorem (F).**

(a) *Let* $\mathbf{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{s}_2 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$, $\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{t}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Take for granted that $\mathbf{s}_1$, $\mathbf{s}_2$ *are linearly independent, and that* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ *constitute a basis for* $\mathbb{R}^3$.

We want to obtain a basis for $\mathbb{R}^3$ *constituted by* $\mathbf{s}_1, \mathbf{s}_2$ *and some vectors from amongst* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$.

Define $U = \left[\, \mathbf{s}_1 \,\middle|\, \mathbf{s}_2 \,\middle|\, \mathbf{t}_1 \,\middle|\, \mathbf{t}_2 \,\middle|\, \mathbf{t}_3 \,\right]$.
We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \left[\begin{array}{cc|ccc} 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 \\ 2 & 6 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow U' = \left[\begin{array}{cc|ccc} 1 & 0 & 3 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -1/2 \end{array}\right]$$

The rank of U' is 3.
The pivot columns of U' are the first, second and fourth columns.

Hence a basis for $\mathbb{R}^3$ is constituted by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{t}_2$.

7. Illustrations for Theorem (F).

(a) *Let* $\mathbf{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{s}_2 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$, $\mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{t}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Take for granted that $\mathbf{s}_1, \mathbf{s}_2$ *are linearly independent, and that* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ *constitute a basis for* $\mathbb{R}^3$.

We want to obtain a basis for $\mathbb{R}^3$ *constituted by* $\mathbf{s}_1, \mathbf{s}_2$ *and some vectors from amongst* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$.

Define $U = \left[\, \overset{u_1}{\mathbf{s}_1} \,\middle|\, \overset{u_2}{\mathbf{s}_2} \,\middle|\, \overset{u_3}{\mathbf{t}_1} \,\middle|\, \overset{u_4}{\mathbf{t}_2} \,\middle|\, \overset{u_5}{\mathbf{t}_3} \,\right]$.

We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{array}{c} \left[\begin{array}{c|c|c|c|c} 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 \\ 2 & 6 & 0 & 0 & 1 \end{array}\right] \\ \begin{array}{ccccc} u_{d_1} & u_{d_2} & & u_{d_3} & \end{array} \end{array} \longrightarrow \cdots\cdots \longrightarrow U' = \begin{array}{c} \left[\begin{array}{c|c|c|c|c} 1 & 0 & 3 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -1/2 \end{array}\right] \\ \begin{array}{ccccc} u'_{d_1} & u'_{d_2} & & u'_{d_3} & \end{array} \end{array}$$

The rank of U' is 3.

The pivot columns of U' are the first, second and fourth columns.

Hence a basis for $\mathbb{R}^3$ is constituted by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{t}_2$.

(b) *Let* $\mathbf{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{t}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix}$, $\mathbf{t}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$. *Define* $W = \mathsf{Span}\left(\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}\right)$.

Take for granted that $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ *are constitutes a basis for* W.

Note that $\mathbf{s}_1$ *is linearly independent. Take for granted that* $\mathbf{s}_1 \in W$.

We want to obtain a basis for W *constituted by* $\mathbf{s}_1$ *and some vectors from amongst* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$.

Define $U = \left[\, \mathbf{s}_1 \,\middle|\, \mathbf{t}_1 \,\middle|\, \mathbf{t}_2 \,\middle|\, \mathbf{t}_3 \,\right]$.
We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \left[\begin{array}{c|ccc} 1 & 2 & 7 & 1 \\ 1 & 1 & 3 & 1 \\ 3 & 2 & 5 & -1 \\ 1 & -1 & -5 & 2 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow U' = \left[\begin{array}{c|ccc} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The rank of U' is 3.
The pivot columns of U' are the first, second and fourth columns.

Hence a basis for W is constituted by $\mathbf{s}_1, \mathbf{t}_1, \mathbf{t}_3$.

(b) *Let* $\mathbf{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{t}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{t}_2 = \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix}$, $\mathbf{t}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$. *Define* $W = \mathsf{Span}\,(\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\})$.

Take for granted that $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ *are constitutes a basis for* W.

Note that $\mathbf{s}_1$ *is linearly independent. Take for granted that* $\mathbf{s}_1 \in W$.

We want to obtain a basis for W *constituted by* $\mathbf{s}_1$ *and some vectors from amongst* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$.

Define $U = \left[\, \mathbf{s}_1 \,\middle|\, \mathbf{t}_1 \,\middle|\, \mathbf{t}_2 \,\middle|\, \mathbf{t}_3 \,\right]$.
We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{array}{c} \left[\begin{array}{c|c|c|c} 1 & 2 & 7 & 1 \\ 1 & 1 & 3 & 1 \\ 3 & 2 & 5 & -1 \\ 1 & -1 & -5 & 2 \end{array}\right] \\ \begin{array}{cccc} u_{d_1} & u_{d_2} & \quad & u_{d_3} \end{array} \end{array} \longrightarrow \cdots\cdots \longrightarrow U' = \begin{array}{c} \left[\begin{array}{c|c|c|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} u'_{d_1} & u'_{d_2} & \quad & u'_{d_3} \end{array} \end{array}$$

The rank of U' is 3.
The pivot columns of U' are the first, second and fourth columns.

Hence a basis for W is constituted by $\mathbf{s}_1, \mathbf{t}_1, \mathbf{t}_3$.

(c) *Let* $\mathbf{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{s}_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{s}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, *and* $\mathbf{t}_j = \mathbf{e}_j^{(5)}$ *for each* $j = 1, 2, 3, 4, 5$.

Take for granted that $\mathbf{s}_1$, $\mathbf{s}_2, \mathbf{s}_3$ *are linearly independent, and that* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5$ *constitute a basis for* $\mathbb{R}^5$.

We want to obtain a basis for $\mathbb{R}^5$ *constituted by* $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ *and some vectors from amongst* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5$.

Define $U = \left[\ \mathbf{s}_1 \,\middle|\, \mathbf{s}_2 \,\middle|\, \mathbf{s}_3 \,\middle|\, \mathbf{t}_1 \,\middle|\, \mathbf{t}_2 \,\middle|\, \mathbf{t}_3 \,\middle|\, \mathbf{t}_4 \,\middle|\, \mathbf{t}_5\ \right]$.

We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \left[\begin{array}{ccc|ccccc} -2 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & -4 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow U' = \left[\begin{array}{ccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 & 4 \end{array}\right]$$

The rank of U' is 5.

The pivot columns of U' are the first, second, third, fourth and fifth columns.

Hence a basis for $\mathbb{R}^5$ is constutitued by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{t}_1, \mathbf{t}_2$.

(c) *Let* $\mathbf{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{s}_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{s}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, *and* $\mathbf{t}_j = \mathbf{e}_j^{(5)}$ *for each* $j = 1, 2, 3, 4, 5$.

Take for granted that $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ *are linearly independent, and that* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5$ *constitute a basis for* $\mathbb{R}^5$.

We want to obtain a basis for $\mathbb{R}^5$ *constituted by* $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ *and some vectors from amongst* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5$.

Define $U = \left[\, \overset{u_1}{\mathbf{s}_1} \,\middle|\, \overset{u_2}{\mathbf{s}_2} \,\middle|\, \overset{u_3}{\mathbf{s}_3} \,\middle|\, \overset{u_4}{\mathbf{t}_1} \,\middle|\, \overset{u_5}{\mathbf{t}_2} \,\middle|\, \overset{u_6}{\mathbf{t}_3} \,\middle|\, \overset{u_7}{\mathbf{t}_4} \,\middle|\, \overset{u_8}{\mathbf{t}_5} \,\right]$.

We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{array}{c} \left[\begin{array}{ccc|ccccc} -2 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & -4 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \\ \begin{array}{cccccccc} u_{d_1} & u_{d_2} & u_{d_3} & u_{d_4} & u_{d_5} & & & \end{array} \end{array} \longrightarrow \cdots\cdots \longrightarrow U' = \begin{array}{c} \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 & 4 \end{array}\right] \\ \begin{array}{cccccccc} u'_{d_1} & u'_{d_2} & u'_{d_3} & u'_{d_4} & u'_{d_5} & & & \end{array} \end{array}$$

The rank of U' is 5.

The pivot columns of U' are the first, second, third, fourth and fifth columns.

Hence a basis for $\mathbb{R}^5$ is constututued by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{t}_1, \mathbf{t}_2$.

(d) *Let*

$$\mathbf{s}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{s}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{s}_4 = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and

$$\mathbf{t}_j = \mathbf{e}_j^{(7)} \text{ for each } j = 1, 2, 3, 4, 5, 6, 7.$$

Take for granted that $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4$ *are linearly independent, and that* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5, \mathbf{t}_6, \mathbf{t}_7$ *constitute a basis for* $\mathbb{R}^7$.

We want to obtain a basis for $\mathbb{R}^7$ *constituted by* $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4$ *and some vectors from amongst* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5, \mathbf{t}_6, \mathbf{t}_7$.

Define $U = \left[\, \mathbf{s}_1 \,\middle|\, \mathbf{s}_2 \,\middle|\, \mathbf{s}_3 \,\middle|\, \mathbf{s}_4 \,\middle|\, \mathbf{t}_1 \,\middle|\, \mathbf{t}_2 \,\middle|\, \mathbf{t}_3 \,\middle|\, \mathbf{t}_4 \,\middle|\, \mathbf{t}_5 \,\middle|\, \mathbf{t}_6 \,\middle|\, \mathbf{t}_7 \,\right]$.

(d) *Let*

$$\mathbf{s}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{s}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{s}_4 = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and

$$\mathbf{t}_j = \mathbf{e}_j^{(7)} \textit{ for each } j = 1, 2, 3, 4, 5, 6, 7.$$

Take for granted that $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4$ *are linearly independent, and that* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5, \mathbf{t}_6, \mathbf{t}_7$ *constitute a basis for* $\mathbb{R}^7$.

We want to obtain a basis for $\mathbb{R}^7$ *constituted by* $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4$ *and some vectors from amongst* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5, \mathbf{t}_6, \mathbf{t}_7$.

Define $U = \left[\, \underset{u_1}{\mathbf{s}_1} \,\middle|\, \underset{u_2}{\mathbf{s}_2} \,\middle|\, \underset{u_3}{\mathbf{s}_3} \,\middle|\, \underset{u_4}{\mathbf{s}_4} \,\middle|\, \underset{u_5}{\mathbf{t}_1} \,\middle|\, \underset{u_6}{\mathbf{t}_2} \,\middle|\, \underset{u_7}{\mathbf{t}_3} \,\middle|\, \underset{u_8}{\mathbf{t}_4} \,\middle|\, \underset{u_9}{\mathbf{t}_5} \,\middle|\, \underset{u_{10}}{\mathbf{t}_6} \,\middle|\, \underset{u_{11}}{\mathbf{t}_7} \,\right]$.

We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \left[\begin{array}{cccc|ccccccc} -4 & -2 & -1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 6 & -6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \cdots\cdots \longrightarrow U' \left[\begin{array}{cccc|ccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -6 & 6 \end{array}\right]$$

The rank of U' is 7.
The pivot columns of U' are the first, second, third, fourth, fifth, seventh and eighth columns.

Hence a basis for W is constituted by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4, \mathbf{t}_1, \mathbf{t}_3, \mathbf{t}_4$.

We find the reduced row-echelon form U' which is row equivalent to U:

$$
\begin{array}{ccccccccccc}
u_{d_1} & u_{d_2} & u_{d_3} & u_{d_4} & u_{d_5} & & u_{d_6} & u_{d_7} & & &
\end{array}
$$

$$
U = \left[\begin{array}{cccc|ccccccc}
-4 & -2 & -1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 3 & -5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -2 & 6 & -6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
\longrightarrow \cdots\cdots \longrightarrow U' \left[\begin{array}{cccc|ccccccc}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 2 & 1 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -3 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -6 & 6
\end{array}\right]
$$

$$
\begin{array}{ccccccccccc}
u'_{d_1} & u'_{d_2} & u'_{d_3} & u'_{d_4} & u'_{d_5} & & u'_{d_6} & u'_{d_7} & & &
\end{array}
$$

The rank of U' is 7.

The pivot columns of U' are the first, second, third, fourth, fifth, seventh and eighth columns.

Hence a basis for W is constituted by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4, \mathbf{t}_1, \mathbf{t}_3, \mathbf{t}_4$.

8. Recall the statement of Theorem (B) from the handout *Bases for subspaces of* $\mathbb{R}^n$:

   Any two bases for a subspace of $\mathbb{R}^n$ *have the same number of vectors.*

   Equipped with Theorem (F), we are now ready to prove Theorem (B).

9. **Proof of Theorem (B).**

   Let W be a subspace of $\mathbb{R}^n$.

   If W be the zero subspace, then the empty set is the only basis for W, and in this situation, there is nothing to prove.

   From now on suppose W is a non-zero subspace of $\mathbb{R}^n$.

   Suppose $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_p$ constitute a basis for W.

   Also suppose $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{p'}$ constitute a basis for W.

   We are going to verify that $p = p'$:

   - By assumption, $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_p$ are linearly independent vectors in W, and $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{p'}$ constitute a basis for W. Then $p \leq p'$.

     Also by assumption, $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{p'}$ are linearly independent vectors in W, and $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_p$ constitute a basis for W. Then $p' \leq p$.

     It follows that $p = p'$.

10. Now recall Theorem (C) below, proved in the handout *Bases for subspaces of* $\mathbb{R}^n$:

> *Suppose* W *is a non-zero subspace of* $\mathbb{R}^n$*. Then* W *has a basis which consists of at least one and at most* n *vectors in* $\mathbb{R}^n$*.*

Combining Theorem (C) and Theorem (F), we will obtain Theorem (G).

11. **Theorem (G). (Extension of linearly independent set to basis in the context of $\mathbb{R}^n$.)**

Let W *be a non-zero subspace of* $\mathbb{R}^n$*, and* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ *be vectors in* W*.*
Further suppose that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ *are linearly independent.*

Then, there is some basis for W*, which is constituted of at most* n *vectors, amongst them being the vectors* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$*.*

Remark.

In plain words, the conclusion in this result says that

> *the linearly independent vectors* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ *in* W *(which do not necessarily constitute a basis for* W *because there may be not enough of them to 'span' every vector in* W*) can be 'extended' to give a basis for* W*.*

12. **Proof of Theorem (G).**

By Theorem (C), W has a basis, constituted by, say, some q vectors

$$\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}$$

in W, for which

$$q \leq n.$$

None of these q vectors is the zero vector.

By assumption:

- none of the p vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ is the zero vectors, and
- each of these vectors is a linear combination of $\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}$.

Then by Theorem (F), we have

$$q \geq p,$$

and there is a basis for W which is constituted by

- $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ together with
- some $q - p$ vectors from amongst $\mathbf{u}_{p+1}, \mathbf{u}_{p+2}, \cdots, \mathbf{u}_{p+q}$.