MATH1030 Gaussian elimination and basis for null space.

1. Refer to the handout *Homogeneous systems and null spaces.* There we learn what to do when we try to give an explicit description for the null space of a matrix:

   Suppose we are given an $(m \times n)$ matrix A.

   To determine $\mathcal{N}(A)$ is the same as giving an 'explicit' description of the solution set of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ through set language, in terms of (hopefully just a few) solutions of the system. That amounts to finding all solutions of $\mathcal{LS}(A, \mathbf{0})$.

   Suppose A' is the reduced row-echelon form which is row-equivalent to A.

   Suppose the rank of A' is r. Write $p = n - r$.

   * When $p = 0$, $\mathcal{N}(A) = \{0\}$.
   * Suppose $p > 0$. Then those (few) solutions $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ of $\mathcal{LS}(A, \mathbf{0})$ needed for expressing all solutions of $\mathcal{LS}(A, \mathbf{0})$ are 'read off' as solutions of $\mathcal{LS}(A', \mathbf{0})$ for which one free variable takes the value 1 and all other free variable take the value 0.
     In conclusion we have

$$\mathcal{N}(A) = \mathcal{N}(A') = \{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p \mid c_1, c_2, \cdots, c_p \in \mathbb{R}\} = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\}).$$

   According to Theorem (D) below, what we are actually doing in this procedure is to find a basis for $\mathcal{N}(A)$. In short, to 'solve' a homogeneous system of linear equations is the same as finding a basis for the null space of the coefficient matrix for the system.

2. **Theorem (D).**

   Let A be an $(m \times n)$-matrix, and A' be the reduced row-echelon form which is row-equivalent to A.

   Suppose the rank of A' is r. Label the pivot columns of A', from left to right, by $d_1, d_2, \cdots, d_r$.

   Write $p = n - r$. Suppose $p > 0$. Label the free columns of A', from left to right, by $f_1, f_2, \cdots, f_p$.

   For each $h = 1, 2, \cdots, r$, and each $k = 1, 2, \cdots, p$, denote by s_{hk} the (d_h, f_k)-th entry of A'.

   For each $k = 1, 2, \cdots, p$, define $\mathbf{u}_k$ to be the vector in $\mathbb{R}^n$ whose f_k-th entry is 1, whose f_j-th entry is 0 whenever $k \neq j$, and whose d_h-th entry is $-s_{hk}$ for each $h = 1, 2, \cdots, r$.

   Then the statements below hold:

   (a) $\mathbf{u}_k \in \mathcal{N}(A)$ *for each* $k = 1, 2, \cdots, p$.

   (b) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ *are linearly independent.*

   (c) *Every vector in* $\mathcal{N}(A)$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$.

   (d) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ *constitute a basis for* $\mathcal{N}(A)$.

   Remark. Once we make sense of the notion of *dimension*, it will turn that the dimension of $\mathcal{N}(A)$ is p, because one base for $\mathcal{N}(A)$, namely, $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ is constituted by p vectors.

3. **Proof of Theorem (D).**

   Suppose $k = 1, 2, \cdots, p$. Denote the ℓ-th entry of $\mathbf{u}_k$ by $u_{k,\ell}$.

   By assumption,

$$u_{k,\ell} = \begin{cases} -s_{hk} & \text{if} \quad \ell = d_h \text{ and } 1 \leq h \leq r \\ 1 & \text{if} \quad \ell = f_k \\ 0 & \text{if} \quad \ell = f_j \text{ and } j \neq k \end{cases}$$

   Denote the (i, ℓ)-th entry of A' by $a'_{i\ell}$.

   (a) • Suppose $i > r$. Then $a'_{i\ell} = 0$ for each ℓ. Therefore the i-th entry of $A'\mathbf{u}_k$ is given by $a'_{i1}u_{k,1} + a'_{i2}u_{k,2} + \cdots + a'_{in}u_{k,n} = 0$.

   • Suppose $i = 1, 2, \cdots, r$. Then

$$a'_{i\ell} = \begin{cases} 1 & \text{if} \quad \ell = d_i \\ 0 & \text{if} \quad \ell = d_h \text{ and } h \neq i \\ s_{ij} & \text{if} \quad \ell = f_j \text{ and } 1 \leq j \leq p \end{cases}$$

   Note that whenever $h \neq i$, we have $a'_{id_h}u_{k,d_h} = 0$. Also, whenever $j \neq k$, we have $a'_{if_j}u_{k,f_j} = 0$.

Hence the i-th entry of $A'\mathbf{u}_k$ is given by

$$\begin{aligned} & a'_{i1}u_{k,1} + a'_{i2}u_{k,2} + \cdots + a'_{in}u_{k,n} \\ = \;& (a'_{id_1}u_{k,d_1} + a'_{id_2}u_{k,d_2} + \cdots + a'_{id_r}u_{k,d_r}) + (a'_{if_1}u_{k,f_1} + a'_{if_2}u_{k,f_2} + \cdots + a'_{if_p}u_{k,f_p}) \\ = \;& a'_{id_i}u_{k,d_i} + a'_{if_k}u_{k,f_k} \\ = \;& 1 \cdot (-s_{ik}) + s_{ik} \cdot 1 = 0. \end{aligned}$$

Therefore $A'\mathbf{u}_k = \mathbf{0}$. It follows that '$\mathbf{x} = \mathbf{u}_k$' is a solution of $\mathcal{LS}(A', \mathbf{0})$, and hence a solution of $\mathcal{LS}(A, \mathbf{0})$ as well. Therefore $\mathbf{u}_k \in \mathcal{N}(A)$.

(b) Pick any $\alpha_1, \alpha_2, \cdots, \alpha_p \in \mathbb{R}$.
Suppose $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots \alpha_p\mathbf{u}_p = \mathbf{0}_n$.
Suppose $j = 1, 2, \cdots, p$. Recall that $u_{j,f_j} = 1$, and $u_{k,f_j} = 0$ whenever $k \neq j$.
Then the f_j-th entry of the vector $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots \alpha_p\mathbf{u}_p$ is given by $\alpha_1 u_{1,f_j} + \alpha_2 u_{2,f_j} + \alpha_p u_{p,f_j} = \alpha_j u_{j,f_j} = \alpha_j$.
The f_j-th entry of $\mathbf{0}_n$ is 0. Therefore $\alpha_j = 0$.
It follows that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent.

(c) Pick any $\mathbf{x} \in \mathcal{N}(A)$. Denote the i-th entry of $\mathbf{x}$ by x_i.
Then $A'\mathbf{x} = \mathbf{0}$. Therefore,

$$\left\{ \begin{array}{lcrcrcccr} x_{d_1} & = & -s_{11}x_{f_1} & - & s_{12}x_{f_2} & - & \cdots & - & s_{1p}x_{f_p} \\ x_{d_2} & = & -s_{21}x_{f_1} & - & s_{22}x_{f_2} & - & \cdots & - & s_{2p}x_{f_p} \\ & \vdots & & & & & & & \\ x_{d_r} & = & -s_{r1}x_{f_1} & - & s_{r2}x_{f_2} & - & \cdots & - & s_{rp}x_{f_p} \end{array} \right. .$$

Therefore $\mathbf{x} = x_{f_1}\mathbf{u}_1 + x_{f_2}\mathbf{u}_2 + \cdots + x_{f_p}\mathbf{u}_p$. (Why?)
It follows that $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$.

(d) According to definition, $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ is a basis for $\mathcal{N}(A)$.

4. **Illustrations of the content of Theorem (D).**

(a) Let $A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix}$.

We obtain the reduced row-echelon form A' which is row-equivalent to A by applying a sequence of row operations to A:

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix} = A'$$

Note that $\mathcal{LS}(A', \mathbf{0})$ reads:

$$\left\{ \begin{array}{ccccccc} x_1 & & & + & 2x_4 & = & 0 \\ & x_2 & & - & 3x_4 & = & 0 \\ & & x_3 & + & 4x_4 & = & 0 \end{array} \right.$$

We have $\mathcal{N}(A) = \{ c\mathbf{u} \mid c \in \mathbb{R} \}$, in which $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ -4 \\ 1 \end{bmatrix}$.

A basis for $\mathcal{N}(A)$ is constituted by the vector $\mathbf{u}$.

(b) Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{bmatrix}$.

We obtain the reduced row-echelon form A' which is row-equivalent to A by applying a sequence of row operations to A:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A'$$

Note that $\mathcal{LS}(A', \mathbf{0})$ reads:

$$\left\{ \begin{array}{ccccccccc} x_1 & & + & 2x_3 & - & 3x_4 & = & 0 \\ & x_2 & - & x_3 & + & 2x_4 & = & 0 \\ & & & & & 0 & = & 0 \end{array} \right.$$

We have

$$\mathcal{N}(A) = \{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 \mid c_1, c_2 \in \mathbb{R}\},$$

in which $\mathbf{u}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$.

A basis for $\mathcal{N}(A)$ is constituted by the vectors $\mathbf{u}_1, \mathbf{u}_2$.

(c) Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix}$.

We obtain the reduced row-echelon form A' which is row-equivalent to A by applying a sequence of row operations to A:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A'$$

Note that $\mathcal{LS}(A', \mathbf{0})$ reads:

$$\begin{cases} x_1 + 2x_3 - 3x_4 - x_5 = 0 \\ x_2 - x_3 + 2x_4 + 4x_5 = 0 \\ 0 = 0 \end{cases}$$

We have

$$\mathcal{N}(A) = \{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 \mid c_1, c_2, c_3 \in \mathbb{R}\},$$

in which $\mathbf{u}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

A basis for $\mathcal{N}(A)$ is constituted by the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

(d) Let $A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}$.

We obtain the reduced row-echelon form A' which is row-equivalent to A by applying a sequence of row operations to A:

$$A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 1 & 2 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A'$$

Note that $\mathcal{LS}(A', \mathbf{0})$ reads:

$$\begin{cases} x_1 + 4x_2 + 2x_5 + x_6 - 3x_7 = 0 \\ x_3 + x_5 - 3x_6 + 5x_7 = 0 \\ x_4 + 2x_5 - 6x_6 + 6x_7 = 0 \\ 0 = 0 \end{cases}$$

We have

$$\mathcal{N}(A) = \{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 \mid c_1, c_2, c_3, c_4 \in \mathbb{R}\},$$

in which $\mathbf{u}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

A basis for $\mathcal{N}(A)$ is constituted by the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.