MATH1030 Further examples on basis for $\mathbb{R}^n$

1. Consider the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n$ of $\mathbb{R}^n$. Determine whether such a collection constitutes a basis for $\mathbb{R}^n$.

   (a) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$.

   (b) $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$.

   (c) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 4 \end{bmatrix}$.

   (d) $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 5 \end{bmatrix}$.

   (e) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 5 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 2 \\ -2 \\ 4 \\ 5 \end{bmatrix}$.

Answer.

Recall the result from the handout *Bases for subspaces of* $\mathbb{R}^n$:

> *Suppose* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are vectors in* $\mathbb{R}^n$*, and* $U$ *is the* $(n \times n)$*-square matrix given by* $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n\ ]$.
> *Then the statements below are logically equivalent:*
>
> (a) $U$ *is non-singular.*
>
> (b) $U$ *is invertible.*
>
> (c) *Every vector in* $\mathbb{R}^n$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.
>
> (d) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent.*
>
> (e) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *constitute a basis for* $\mathbb{R}^n$.

1. (a) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is not a basis for $\mathbb{R}^3$ because $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent.
   Reason:

$$[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$$

   (b) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is a basis for $\mathbb{R}^3$ because $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent.
   Reason:

$$[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$$

   (c) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ is not a basis for $\mathbb{R}^4$ because $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly dependent.
   Reason:

$$[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4\ ] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   (d) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ is a basis for $\mathbb{R}^4$ because $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly independent.
   Reason:

$$[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4\ ] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   (e) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ is not a basis for $\mathbb{R}^4$ because $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly dependent.
   Reason:

$$[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4\ ] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$