1. **Definition. (Linear dependence and linear independence.)**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.

(a) *We say that* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly dependent if and only if the statement* (LD) *holds:*

(LD) *There exist some* $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$ *such that*

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$

and

$$\alpha_1, \alpha_2, \cdots, \alpha_n \text{ are not all zero.}$$

The equality

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$

in which $\alpha_1, \alpha_2, \cdots, \alpha_n$ *are not all zero is called a non-trivial linear relation of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

(b) *We say that* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent if and only if*

it is not true that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly dependent.*

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$\alpha_1, \alpha_2, \cdots, \alpha_n$ *are not all zero.*

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it is not true that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly dependent.*

2. **Illustrations.**

(a) Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 3 \\ 6 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 4 \\ 5 \\ 6 \\ 3 \\ 4 \end{bmatrix}$.

The non-trivial linear relation $2\mathbf{u}_1 - 3\mathbf{u}_2 + 4\mathbf{u}_3 - 1 \cdot \mathbf{u}_4 = \mathbf{0}$ holds.

Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly dependent.

(b) Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 3 \\ -2 \\ -6 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} -3 \\ -1 \\ 1 \\ -3 \\ 1 \end{bmatrix}$.

The non-trivial linear relation $3\mathbf{u}_1 + 5\mathbf{u}_2 + 3\mathbf{u}_3 + 2\mathbf{u}_4 + 4\mathbf{u}_5 = \mathbf{0}$ holds.

Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ are linearly dependent.

3. **Comment on the definition of linear dependence and the definition of linear independence.**

   The above version of the definition for the notion of *linear independence* is not ideal.

   We only know that being *linear independent* is ‘exactly opposite’ to being *linear dependent*.

   But this will not be helpful in practice (whether in theoretical discussions or in computations).

   The situation will be improved once we find some re-formulations for the notion of *linear dependence*, and apply them to give much more useful re-formulations for the notion of *linear independence*.

4. Recall Lemma (A):

Let A be an $(m \times n)$-matrix, and $\mathbf{t}$ be a vector in $\mathbb{R}^n$.

Suppose that for each $j = 1, 2, \cdots, n$, the j-th column of A is $\mathbf{a}_j$ and the j-th entry of $\mathbf{t}$ is t_j. (So $A = \left[\, \mathbf{a}_1 \,\middle|\, \mathbf{a}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{a}_n \,\right]$ and $\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$.) Then $A\mathbf{t} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n$.

By applying Lemma (A), we obtain Theorem (G) below.

5. **Theorem (G). (Re-formulations for the notion of linear dependence.)**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^m$. Define the $(m \times n)$-matrix U by $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{u}_n \,\right]$.

The statements below are logically equivalent:

($\diamondsuit$) *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly dependent.*

($\clubsuit$) *The homogeneous system of linear equations $\mathcal{LS}(U, \mathbf{0}_m)$ has a non-trivial solution.*

($\heartsuit$) *$\mathcal{N}(U) \neq \{\mathbf{0}_n\}$.*

($\spadesuit$) *There exists some $\mathbf{t} \in \mathbb{R}^n$ such that $\mathbf{t} \in \mathcal{N}(U)$ and $\mathbf{t} \neq \mathbf{0}_n$.*

4. Recall Lemma (A):

Let A be an $(m \times n)$-matrix, and $\mathbf{t}$ be a vector in $\mathbb{R}^n$.

Suppose that for each $j = 1, 2, \cdots, n$, the j-th column of A is $\mathbf{a}_j$ and the j-th entry of $\mathbf{t}$ is t_j. (So $A = \left[\, \mathbf{a}_1 \middle| \mathbf{a}_2 \middle| \cdots \middle| \mathbf{a}_n \,\right]$ and $\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$.) Then $A\mathbf{t} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n$.

By applying Lemma (A), we obtain Theorem (G) below.

'Translation' of 'linear dependence' into various things that we are more familiar.

5. **Theorem (G). (Re-formulations for the notion of linear dependence.)**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^m$. Define the $(m \times n)$-matrix U by $U = \left[\, \mathbf{u}_1 \middle| \mathbf{u}_2 \middle| \cdots \middle| \mathbf{u}_n \,\right]$.

The statements below are logically equivalent:

($\diamondsuit$) *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly dependent.* ← Not so familiar at this moment.

($\clubsuit$) *The homogeneous system of linear equations $\mathcal{LS}(U, \mathbf{0}_m)$ has a non-trivial solution.*

($\heartsuit$) $\mathcal{N}(U) \neq \{\mathbf{0}_n\}$.

($\spadesuit$) *There exists some $\mathbf{t} \in \mathbb{R}^n$ such that $\mathbf{t} \in \mathcal{N}(U)$ and $\mathbf{t} \neq \mathbf{0}_n$.*

More familiar and more 'usable'.

6. **Proof of Theorem (G).**

By the definition of null space, the statements (♣), (♡), (♠) are logically equivalent.

We verify that the statements (♢), (♣) are logically equivalent:

- Suppose (♢) holds.

  Then there exist some $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$ such that $\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}_m$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ are not all zero.

  Define $\mathbf{t} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$. Since $\alpha_1, \alpha_2, \cdots, \alpha_n$ are not all zero, $\mathbf{t}$ is not the zero vector in $\mathbb{R}^n$.

  By Lemma (A), we have $U\mathbf{t} = \mathbf{0}_m$. Then '$\mathbf{x} = \mathbf{t}$' is a non-trivial solution of $\mathcal{LS}(U, \mathbf{0}_m)$.

  Hence (♣) holds.

- Suppose (♣) holds. Then $\mathcal{LS}(U, \mathbf{0}_m)$ has a non-trivial solution, say, '$\mathbf{x} = \mathbf{t}$'.

  Denote the j-th entry of $\mathbf{t}$ by α_j for each j.

  Since $\mathbf{t}$ is not the zero vector in $\mathbb{R}^n$, $\alpha_1, \alpha_2, \cdots, \alpha_n$ are not all zero.

  Now by Lemma (A), we have $\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n = U\mathbf{t} = \mathbf{0}_m$.

  Hence (♢) holds.

7. **Corollary to Theorem (G). (Re-formulation for the notion of linear independence.)**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.

Define the $(m \times n)$*-matrix* U *by* $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{u}_n \,\right]$.

The statements below are logically equivalent:

($\sim\diamondsuit$) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent.*

($\sim\clubsuit$) *The only solution of homogeneous system of linear equations* $\mathcal{LS}(U, \mathbf{0}_m)$ *is the trivial solution.*

($\sim\heartsuit$) $\mathcal{N}(U) = \{\mathbf{0}_n\}$.

($\sim\spadesuit$) *For any* $\mathbf{t} \in \mathbb{R}^n$, *if* $\mathbf{t} \in \mathcal{N}(U)$ *then* $\mathbf{t} = \mathbf{0}_n$.

Remark.

($\sim\clubsuit$), ($\sim\spadesuit$) are better re-formulations for the notion of linear independence in the sense that they are easiest to use.

($\sim\spadesuit$) can be further re-formulated into something even better.

The logical equivalence amongst ($\sim\diamondsuit$), ($\sim\clubsuit$), ($\sim\heartsuit$), ($\sim\spadesuit$) comes from the 'fact' that 'blah-blah-blah being linearly independent' is the same as 'blah-blah-blah being not linearly dependent'.

7. **Corollary to Theorem (G). (Re-formulation for the notion of linear independence.)**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.

Define the $(m \times n)$*-matrix* U *by* $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{u}_n \,\right]$.

The statements below are logically equivalent:

Not so familiar at this moment.

($\sim\diamondsuit$) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent.* ($u_1, u_2, \ldots, u_n$ are not linearly dependent.)

($\sim\clubsuit$) *The only solution of homogeneous system of linear equations* $\mathcal{LS}(U, \mathbf{0}_m)$ *is the trivial solution.*

($\sim\heartsuit$) $\mathcal{N}(U) = \{\mathbf{0}_n\}$.

More familiar and more 'usable'.

($\sim\spadesuit$) *For any* $\mathbf{t} \in \mathbb{R}^n$, *if* $\mathbf{t} \in \mathcal{N}(U)$ *then* $\mathbf{t} = \mathbf{0}_n$.

'Translation' of 'linear independence' into various things that we are more 'familiar', through the use of the various versions of re-formulations of 'linear dependence'.

Remark.

($\sim\clubsuit$), ($\sim\spadesuit$) are better re-formulations for the notion of linear independence in the sense that they are easiest to use.

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The logical equivalence amongst ($\sim\diamondsuit$), ($\sim\clubsuit$), ($\sim\heartsuit$), ($\sim\spadesuit$) comes from the 'fact' that 'blah-blah-blah being linearly independent' is the same as 'blah-blah-blah being not linearly dependent'.

8. Theorem (1) is an immediate consequence of Theorem (G).

Theorem (1).

Suppose $m < n$, *and* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are vectors in* $\mathbb{R}^m$.

Then $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly dependent.*

Remark.

In plain words, what Theorem (1) says is that

any $m + 1$ *or more vectors in* $\mathbb{R}^m$ *are definitely linearly dependent.*

9. We very often like to re-formulate Theorem (1) in the form of the statements in Corollary to Theorem (1).

Corollary to Theorem (1).

The statements below hold:

(a) *For each positive integer* k, *any* $m + k$ *vectors in* $\mathbb{R}^m$ *are linearly dependent.*

(b) *Let* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_\ell$ *be vectors in* $\mathbb{R}^m$.

Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_\ell$ *are linearly independent. Then* $\ell \leq m$.

Remark.

After introducing the notions of *basis* and *dimension*, we will establish some results generalize Theorem (1).

10. **Proof of Theorem (1).**

Suppose $m < n$, and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are vectors in $\mathbb{R}^m$.

Define the $(m \times n)$-matrix U by $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{u}_n \,\right]$.

Denote by U' the reduced row-echelon form which is row-equivalent to U.
Note that U' is an $(m \times n)$-matrix.

Since $m < n$, there is at least one free column in U'.

Therefore $\mathcal{LS}(U', \mathbf{0})$ (and hence $\mathcal{LS}(U, \mathbf{0})$) has a non-trivial solution in $\mathbb{R}^n$.

It follows that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly dependent.

11. Theorem (2) is also an immediate consequence of Theorem (G).

Theorem (2).

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are vectors in* $\mathbb{R}^n$, *and* U *is the* $(n \times n)$*-square matrix given by* $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{u}_n \,\right]$.

Then the statements below are logically equivalent:

(a) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent.*

(b) U *is non-singular.*

(c) U *is invertible.*

Remark.

This result can be merged with Theorem (E) in the Handout *Existence and uniqueness of solutions for a system of linear equations whose coefficient matrix is a square matrix* . We will do it later, alongside more re-formulations for the notion of *non-singularity*.

12. **Question.**

Suppose the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *in* $\mathbb{R}^m$ *are given to us in 'concrete' terms.*

How to determine whether $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly dependent or not?*

Answer to the question.

Theorem (G) and Corollary to Theorem (G) have already provided an answer to this question, in terms of some system of linear equations determined by these vectors.

This answer can be put to use in calculations, through an 'algorithm' described below.

13. **'Algorithm' associated to Theorem (G) and Corollary to Theorem (G).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathbb{R}^m$. We are going to determine whether $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly dependent or not:

- **Step (0).**

  Inspect whether $m < n$ or not.

  $*$ If '*yes*', $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are definitely linearly dependent (according to Theorem (1)).

  $*$ If '*no*', proceed to Step (1).

- **Step (1).**

  (From now on, it is assumed that $m \geq n$.) Form the matrix $U = \left[\ \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{u}_n\ \right]$.

- **Step (2).**

  Obtain the reduced row-echelon form U' which is row equivalent to U.

- **Step (3).**

  Read off from U' the answer for the question: '*Does the homogeneous system $\mathcal{LS}(U, \mathbf{0})$ have any non-trivial solution in $\mathbb{R}^n$?*'

  $*$ If '*no*', conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent.

  $*$ If '*yes*', conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly dependent.

  Furthermore, if '$\mathbf{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$' is a non-trivial solution of $\mathcal{LS}(U, \mathbf{0})$ then conclude that

  $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$ is a non-trivial linear relation of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

13. **'Algorithm' associated to Theorem (G) and Corollary to Theorem (G).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathbb{R}^m$. We are going to determine whether $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly dependent or not:

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  * If '*yes*', $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are definitely linearly dependent (according to Theorem (1)).
  * If '*no*', proceed to Step (1).

- **Step (1).**

  (From now on, it is assumed that $m \geq n$.) Form the matrix $U = \left[\begin{array}{c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{array}\right]$.

- **Step (2).**

  Obtain the reduced row-echelon form U' which is row equivalent to U.

  $U \to \cdots \to U'$ ← Reduced row-echelon form.

- **Step (3).**

  Read off from U' the answer for the question: '*Does the homogeneous system $\mathcal{LS}(U, \mathbf{0})$ have any non-trivial solution in $\mathbb{R}^n$?*'

  * If '*no*', conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent. (Every column of U' is a pivot column.)
  * If '*yes*', conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly dependent. (Some column of U' is not a pivot column.)

  Furthermore, if '$\mathbf{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$' is a non-trivial solution of $\mathcal{LS}(U, \mathbf{0})$ then conclude that

  $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$ is a non-trivial linear relation of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

14. **Illustrations.**

(a) *Let* $\mathbf{u}_1 = \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix}$.

We want to determine whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *are linearly dependent or not.*

Define $U = \left[\ \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \ \right]$.

We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\left[\ U' \,\middle|\, \mathbf{0} \ \right]$ is the augmented matrix representation of the homogeneous system $\begin{cases} x_1 & & & = 0 \\ & x_2 & & = 0 \\ & & x_3 & = 0 \end{cases}$.

The homogeneous system $\mathcal{LS}(U', \mathbf{0})$ and hence $\mathcal{LS}(U, \mathbf{0})$ has no non-trivial solution.

Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linear independent.

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We want to determine whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *are linearly dependent or not.*

Define $U = \left[\, \mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \mathbf{u}_3 \,\right]$.

We find the reduced row-echelon form U' which is row equivalent to U:

All columns of U' are pivot columns.

$$U = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\left[\, U' \,|\, \mathbf{0} \,\right]$ is the augmented matrix representation of the homogeneous system $\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$.

The homogeneous system $\mathcal{LS}(U', \mathbf{0})$ and hence $\mathcal{LS}(U, \mathbf{0})$ has no non-trivial solution.

Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linear independent.

(b) *Let* $\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -2 \\ 3 \\ -12 \end{bmatrix}$.

We want to determine whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *are linearly dependent or not.*

Define $U = \left[\mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \right]$.

We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{bmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 2 & 7 & -12 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\left[U' \,\middle|\, \mathbf{0} \right]$ is the augmented matrix representation of the homogeneous system $\left\{ \begin{array}{rcrcl} x_1 & & + & x_3 & = 0 \\ & x_2 & - & 2x_3 & = 0 \\ & & & 0 & = 0 \end{array} \right.$.

The homogeneous system $\mathcal{LS}(U', \mathbf{0})$ (and hence $\mathcal{LS}(U, \mathbf{0})$) with unknown $\mathbf{x}$ in $\mathbb{R}^3$ has some non-trivial solution, for instance, $\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linear dependent, with a non-trivial linear relation, say, $-1 \cdot \mathbf{u}_1 + 2\mathbf{u}_2 + 1 \cdot \mathbf{u}_3 = \mathbf{0}$.

(b) *Let* $\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -2 \\ 3 \\ -12 \end{bmatrix}$.

We want to determine whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *are linearly dependent or not.*

Define $U = \left[\, \mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \mathbf{u}_3 \,\right]$.

We find the reduced row-echelon form U' which is row equivalent to U:

Some column, such as the last column, is not a pivot column.

$$U = \begin{bmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 2 & 7 & -12 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\left[\, U' \,|\, \mathbf{0} \,\right]$ is the augmented matrix representation of the homogeneous system $\begin{cases} x_1 \quad + \quad x_3 = 0 \\ \quad x_2 - 2x_3 = 0 \\ \qquad\quad 0 = 0 \end{cases}$.

The homogeneous system $\mathcal{LS}(U', \mathbf{0})$ (and hence $\mathcal{LS}(U, \mathbf{0})$) with unknown $\mathbf{x}$ in $\mathbb{R}^3$ has some non-trivial solution, for instance, $\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linear dependent, with a non-trivial linear relation, say, $-1 \cdot \mathbf{u}_1 + 2\mathbf{u}_2 + 1 \cdot \mathbf{u}_3 = \mathbf{0}$.

(c) *Let* $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} -6 \\ 7 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

We want to determine whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *are linearly dependent or not.*

Define $U = \left[\ \mathbf{u}_1 \ \middle| \ \mathbf{u}_2 \ \middle| \ \mathbf{u}_3 \ \middle| \ \mathbf{u}_4 \ \right]$.

We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{bmatrix} 2 & 1 & 2 & -6 \\ -1 & 2 & 1 & 7 \\ 3 & -1 & -3 & -1 \\ 1 & 5 & 6 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\left[\ U' \ \middle| \ \mathbf{0} \ \right]$ is the augmented matrix representation of the homogeneous system $\begin{cases} x_1 & & & & = 0 \\ & x_2 & & & = 0 \\ & & x_3 & & = 0 \\ & & & x_4 & = 0 \\ & & & 0 & = 0 \end{cases}$.

The homogeneous system $\mathcal{LS}(U', \mathbf{0})$ and hence $\mathcal{LS}(U, \mathbf{0})$ has no non-trivial solution.

Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linear independent.

(c) *Let* $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} -6 \\ 7 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

We want to determine whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *are linearly dependent or not.*

Define $U = \left[\mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\middle|\, \mathbf{u}_4 \right]$.

We find the reduced row-echelon form U' which is row equivalent to U:

All columns are pivot columns.

$$U = \begin{bmatrix} 2 & 1 & 2 & -6 \\ -1 & 2 & 1 & 7 \\ 3 & -1 & -3 & -1 \\ 1 & 5 & 6 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\left[U' \,\middle|\, \mathbf{0} \right]$ is the augmented matrix representation of the homogeneous system $\left\{ \begin{aligned} x_1 \qquad\qquad &= 0 \\ x_2 \qquad &= 0 \\ x_3 \quad &= 0 \\ x_4 &= 0 \\ 0 &= 0 \end{aligned} \right.$.

The homogeneous system $\mathcal{LS}(U', \mathbf{0})$ and hence $\mathcal{LS}(U, \mathbf{0})$ has no non-trivial solution.

Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linear independent.

(d) *Let* $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} -6 \\ 7 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

We want to determine whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *are linearly dependent or not.*

Define $U = \left[\begin{array}{c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{array}\right]$. We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{bmatrix} 2 & 1 & 2 & -6 \\ -1 & 2 & 1 & 7 \\ 3 & -1 & -3 & -1 \\ 1 & 5 & 6 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\left[\begin{array}{c|c} U' & \mathbf{0} \end{array}\right]$ is the augmented matrix representation of the homogeneous system $\left\{\begin{array}{rcrcrcrcl} x_1 & & & & & - & 2x_4 & = & 0 \\ & & x_2 & & & + & 4x_4 & = & 0 \\ & & & & x_3 & - & 3x_4 & = & 0 \\ & & & & & & 0 & = & 0 \\ & & & & & & 0 & = & 0 \end{array}\right.$.

The homogeneous system $\mathcal{LS}(U', \mathbf{0})$ (and hence $\mathcal{LS}(U, \mathbf{0})$) with unknown $\mathbf{x}$ in $\mathbb{R}^4$ has some non-trivial solution, for instance, $\mathbf{x} = \begin{bmatrix} 2 \\ -4 \\ 3 \\ 1 \end{bmatrix}$.

Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linear dependent, with a non-trivial linear relation, say, $2\mathbf{u}_1 - 4\mathbf{u}_2 + 3\mathbf{u}_3 + 1 \cdot \mathbf{u}_4 = \mathbf{0}$.

(d) *Let* $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} -6 \\ 7 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

We want to determine whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ *are linearly dependent or not.*

Define $U = \left[\, \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \,\right]$. We find the reduced row-echelon form U' which is row equivalent to U:

$$U = \begin{bmatrix} 2 & 1 & 2 & -6 \\ -1 & 2 & 1 & 7 \\ 3 & -1 & -3 & -1 \\ 1 & 5 & 6 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow U' = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Some column, such as the last column, is not a pivot column.

$\left[\, U' \mid \mathbf{0} \,\right]$ is the augmented matrix representation of the homogeneous system

$$\begin{cases} x_1 \qquad\qquad\qquad - 2x_4 = 0 \\ \qquad x_2 \qquad\quad + 4x_4 = 0 \\ \qquad\qquad x_3 - 3x_4 = 0 \\ \qquad\qquad\qquad\quad\ 0 = 0 \\ \qquad\qquad\qquad\quad\ 0 = 0 \end{cases}.$$

The homogeneous system $\mathcal{LS}(U', \mathbf{0})$ (and hence $\mathcal{LS}(U, \mathbf{0})$) with unknown $\mathbf{x}$ in $\mathbb{R}^4$ has some non-trivial solution, for instance, $\mathbf{x} = \begin{bmatrix} 2 \\ -4 \\ 3 \\ 1 \end{bmatrix}$.

Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linear dependent, with a non-trivial linear relation, say, $2\mathbf{u}_1 - 4\mathbf{u}_2 + 3\mathbf{u}_3 + 1 \cdot \mathbf{u}_4 = \mathbf{0}$.