1. Recall the definition for the notion of *linear combination*:

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.
Let $\mathbf{v}$ *be a vector in* $\mathbb{R}^m$.
We say $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *if the statement* $(\dagger)$ *holds:*
$(\dagger)$ *There exist some real numbers* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *such that*

$$\mathbf{v} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n.$$

The expression $\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n$ *on its own is called the linear combination of the vectors* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *and the scalars* $\alpha_1, \alpha_2, \cdots, \alpha_n$.

Also recall Lemma (A):

Let A *be an* $(m \times n)$*-matrix, and* $\mathbf{t}$ *be a vector in* $\mathbb{R}^n$.
Suppose that for each $j = 1, 2, \cdots, n$*, the* j*-th column of* A *is* $\mathbf{a}_j$ *and the* j*-th entry of* $\mathbf{t}$ *is* t_j. *(So* $A = [\, \mathbf{a}_1 \,|\, \mathbf{a}_2 \,|\, \cdots \,|\, \mathbf{a}_n \,]$ *and* $\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$*.)*

Then $A\mathbf{t} = t_1 \mathbf{a}_1 + t_2 \mathbf{a}_2 + \cdots + t_n \mathbf{a}_n$.

2. **Question.**

Suppose the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ in $\mathbb{R}^m$ are given to us in 'concrete' terms.

How to determine whether $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ or not?

How do we approach this question?

Lemma (A) will be instrumental in answering this question.

With the help of Lemma (A), we are going to translate this question about a collection of vectors into a question about a system of linear equations determined by these vectors.

We can answer the latter question immediately and completely.

Then with the help of Lemma (A) again, we will translate the answer to the latter question back into a complete answer to the original question.

Answer to the question.

This is provided by Theorem (F) and Corollary to Theorem (F).

2. **Question.**

Suppose the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ in $\mathbb{R}^m$ are given to us in 'concrete' terms.

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With the help of Lemma (A), we are going to translate this question about a collection of vectors into a question about a system of linear equations determined by these vectors.

We can answer the latter question immediately and completely.

Then with the help of Lemma (A) again, we will translate the answer to the latter question back into a complete answer to the original question.

This is a recurrent feature: whenever we encounter a question we do not know how to answer, we 'translate' it into some question that we can answer, and then 'translate' the answer back into one for the original question.

Answer to the question.

This is provided by Theorem (F) and Corollary to Theorem (F).

3. **Theorem (F).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v} \in \mathbb{R}^m$, *and* $U = [\,\mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \cdots \,|\, \mathbf{u}_n\,]$.

The statements below are logically equivalent:

($\sharp$) $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *and scalars* $\alpha_1, \alpha_2, \cdots, \alpha_n$.

($\flat$) *The system* $\mathcal{LS}(U, \mathbf{v})$ *with unknown* $\mathbf{x}$ *is consistent, with a solution '*$\mathbf{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$*'.*

Remark.

When we don't mention the α_j's, Theorem (F) gives:

- $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *if and only if the system* $\mathcal{LS}(U, \mathbf{v})$ *with unknown* $\mathbf{x}$ *is consistent.*

Corollary to Theorem (F).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v} \in \mathbb{R}^m$, *and* $U = [\,\mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \cdots \,|\, \mathbf{u}_n\,]$.

The statements below are logically equivalent:

($\sim\sharp$) $\mathbf{v}$ *is not a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

($\sim\flat$) *The system* $\mathcal{LS}(U, \mathbf{v})$ *is inconsistent.*

4. **Proof of Theorem (F).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v} \in \mathbb{R}^m$, and $U = [\, \mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \cdots \,|\, \mathbf{u}_n \,]$.

- Suppose ($\sharp$) holds. Then $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

  [We want to deduce:

  'The system $\mathcal{LS}(U, \mathbf{v})$ with unknown $\mathbf{x}$ is consistent, with a solution '$\mathbf{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$'.']

  By assumption, $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{v}$.

  Define $\mathbf{t} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$.

  Then by Lemma (A), $U\mathbf{t} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{v}$.

  Therefore '$\mathbf{x} = \mathbf{t}$' is a solution of the system $\mathcal{LS}(U, \mathbf{v})$.

  By definition, $\mathcal{LS}(U, \mathbf{v})$ is consistent.

  Hence ($\flat$) holds.

4. **Proof of Theorem (F).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v} \in \mathbb{R}^m$, and $U = [\, \mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \cdots \,|\, \mathbf{u}_n \,]$.

- Suppose ($\sharp$) holds. Then $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

  [We want to deduce:

  'The system $\mathcal{LS}(U, \mathbf{v})$ with unknown $\mathbf{x}$ is consistent, with a solution '$\mathbf{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$'.]

  By assumption, $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{v}$.

  Can the left-hand side be re-expressed in terms of U?

  Define $\mathbf{t} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$.

  Then by Lemma (A), $U\mathbf{t} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{v}$.

  Therefore '$\mathbf{x} = \mathbf{t}$' is a solution of the system $\mathcal{LS}(U, \mathbf{v})$.

  By definition, $\mathcal{LS}(U, \mathbf{v})$ is consistent.

  Hence ($\flat$) holds.

- Suppose ($\flat$) holds.

  Then the system $\mathcal{LS}(U, \mathbf{v})$ with unknown $\mathbf{x}$ is consistent, with a solution '$\mathbf{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$'.

  [We want to deduce:
  '$\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.']

  Define the vector $\mathbf{t}$ by $\mathbf{t} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$.

  By definition, $\mathbf{v} = U\mathbf{t}$.

  Then, by Lemma (A), $\mathbf{v} = U\mathbf{t} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n$.

  Therefore, by definition, $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

  Hence ($\sharp$) holds.

- Suppose ($\flat$) holds.

Then the system $\mathcal{LS}(U, \mathbf{v})$ with unknown $\mathbf{x}$ is consistent, with a solution '$\mathbf{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$'.

[We want to deduce:
'$\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.']

Define the vector $\mathbf{t}$ by $\mathbf{t} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$.

By definition, $\boxed{\mathbf{v} = U\mathbf{t}.}$ Can the right-hand side be re-expressed in terms of $u_1, u_2, \ldots, u_n$?

Then, by Lemma (A), $\mathbf{v} = U\mathbf{t} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n$.

Therefore, by definition, $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

Hence ($\sharp$) holds.

5. Theorem (F) and Corollary to Theorem (F), combined with what we know about solving equations, suggest an 'algorithm' for determining whether a 'concretely' given vector is a linear combination of a 'concretely' given collection of vectors.

'Algorithm' associated to Theorem (F) and Corollary to Theorem (F).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v} \in \mathbb{R}^m$. We are going to determine whether $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$:

- **Step (1).**

  Form the matrix $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{u}_n \,\right]$.

  Then form the augmented matrix representation $C = \left[\, U \,\middle|\, \mathbf{v} \,\right]$ for the system $\mathcal{LS}(U,\, \mathbf{v})$.

- **Step (2).**

  Obtain some row-echelon form $C^\sharp$ which is row-equivalent to C.

- **Step (3).**

  Inspect $C^\sharp$, to ask whether the last column of $C^\sharp$ contains a leading one.

  $*$ If *yes*, then conclude that $\mathcal{LS}(U,\, \mathbf{v})$ is inconsistent, and further conclude that not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

  $*$ If *no*, then conclude that $\mathcal{LS}(U,\, \mathbf{v})$ is consistent, and further conclude that $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

  To indeed express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$, further obtain (from $C^\sharp$) the reduced row-echelon form C' which is row equivalent to C.

  Read off from C' a solution of $\mathcal{LS}(U,\, \mathbf{v})$.

  If '$\mathbf{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$' is a solution of $\mathcal{LS}(U,\, \mathbf{v})$ then conclude that $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n$.

5. Theorem (F) and Corollary to Theorem (F), combined with what we know about solving equations, suggest an 'algorithm' for determining whether a 'concretely' given vector is a linear combination of a 'concretely' given collection of vectors.

'Algorithm' associated to Theorem (F) and Corollary to Theorem (F).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v} \in \mathbb{R}^m$. We are going to determine whether $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$:

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- **Step (2).**

  Obtain some row-echelon form $C^\sharp$ which is row-equivalent to C.

  $$[U|v] = C \to \cdots \to \underbrace{C^{\#}}_{\text{Some row-echelon form}} = [\underbrace{U^{\#}}_{\text{First } n \text{ columns of } C^{\#}} | \underbrace{v^{\#}}_{\text{Last column of } C^{\#}}]$$

- **Step (3).**

  Inspect $C^\sharp$, to ask whether the last column of $C^\sharp$ contains a leading one.

  * If *yes*, then conclude that $\mathcal{LS}(U, \mathbf{v})$ is inconsistent, and further conclude that not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.
  * If *no*, then conclude that $\mathcal{LS}(U, \mathbf{v})$ is consistent, and further conclude that $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

    To indeed express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$, further obtain (from $C^\sharp$) the reduced row-echelon form C' which is row equivalent to C.

    Read off from C' a solution of $\mathcal{LS}(U, \mathbf{v})$.

    $$[U|v] = C \to \cdots \to C^{\#} \to \cdots \to \underbrace{C'}_{\text{Reduced row-echelon form}} = \underbrace{[\underbrace{U'}_{\text{First } n \text{ columns of } C'} | \underbrace{v'}_{\text{Last column of } C'}]}$$

    If '$\mathbf{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$' is a solution of $\mathcal{LS}(U, \mathbf{v})$ then conclude that $\underline{\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n}$.

    Read off from $\mathcal{LS}(U', v')$ a solution '$x = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$'.

6. **Illustrations.**

(a) *Let* $\mathbf{u}_1 = \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}$.

We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Define $U = \left[\begin{array}{c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{array} \right]$.

The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is $C = \left[\begin{array}{ccc|c} -7 & -6 & -12 & -33 \\ 5 & 5 & 7 & 24 \\ 1 & 0 & 4 & 5 \end{array} \right]$.

We find some row-echelon form and then the reduced row-echelon form C' which are row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C' = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array} \right],$$

which is the augmented matrix representation of the system $\begin{cases} x_1 = -3 \\ x_2 = 5 \\ x_3 = 2 \end{cases}$

Hence $\mathcal{LS}(U, \mathbf{v})$ with unknown $\mathbf{x}$ in $\mathbb{R}^3$ has some solution, namely $\mathbf{x} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$.

Then $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. In fact, $\mathbf{v} = -3\mathbf{u}_1 + 5\mathbf{u}_2 + 2\mathbf{u}_3$.

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We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Define $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\right]$.

The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is $C = \left[\begin{array}{ccc|c} -7 & -6 & -12 & -33 \\ 5 & 5 & 7 & 24 \\ 1 & 0 & 4 & 5 \end{array}\right]$.

We find some row-echelon form and then the reduced row-echelon form C' which are row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C' = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right],$$

There is no leading one in the last column.

which is the augmented matrix representation of the system $\begin{cases} x_1 & & & = & -3 \\ & x_2 & & = & 5 \\ & & x_3 & = & 2 \end{cases}$

Hence $\mathcal{LS}(U, \mathbf{v})$ with unknown $\mathbf{x}$ in $\mathbb{R}^3$ has some solution, namely $\mathbf{x} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$.

Then $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. In fact, $\mathbf{v} = -3\mathbf{u}_1 + 5\mathbf{u}_2 + 2\mathbf{u}_3$.

(b) *Let* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}$ *be vectors in* $\mathbb{R}^3$, *given by* $\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -2 \\ 3 \\ -12 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -4 \\ 11 \end{bmatrix}$.

We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Define $U = \left[\, \mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \mathbf{u}_3 \,\right]$.

The augmented matrix representation of $\mathcal{LS}(U,\, \mathbf{v})$ is $C = \left[\begin{array}{ccc|c} 0 & 1 & -2 & 1 \\ -1 & -2 & 3 & -4 \\ 2 & 7 & -12 & 11 \end{array}\right]$.

We find some row-echelon form and then the reduced row-echelon form C' which are row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C' = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is the augmented matrix representation of the system $\left\{ \begin{array}{rcrcrcl} x_1 & & & + & x_3 & = & 2 \\ & & x_2 & - & 2x_3 & = & 1 \\ & & & & 0 & = & 0 \end{array} \right.$.

Hence $\mathcal{LS}(U,\, \mathbf{v})$ with unknown $\mathbf{x}$ in $\mathbb{R}^3$ has some solution, for instance, $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Then $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. In fact, $\mathbf{v} = 2\mathbf{u}_1 + 1 \cdot \mathbf{u}_2 + 0 \cdot \mathbf{u}_3$.

(b) *Let* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}$ *be vectors in* $\mathbb{R}^3$, *given by* $\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -2 \\ 3 \\ -12 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -4 \\ 11 \end{bmatrix}$.

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Define $U = \left[\, \mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \mathbf{u}_3 \, \right]$.

The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is $C = \left[\begin{array}{ccc|c} 0 & 1 & -2 & 1 \\ -1 & -2 & 3 & -4 \\ 2 & 7 & -12 & 11 \end{array}\right]$.

We find some row-echelon form and then the reduced row-echelon form C' which are row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C' = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There is no leading one in the last column.

which is the augmented matrix representation of the system $\left\{ \begin{array}{rcrcrcl} x_1 & & & + & x_3 & = & 2 \\ & & x_2 & - & 2x_3 & = & 1 \\ & & & & 0 & = & 0 \end{array} \right.$.

Hence $\mathcal{LS}(U, \mathbf{v})$ with unknown $\mathbf{x}$ in $\mathbb{R}^3$ has some solution, for instance, $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Then $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. In fact, $\mathbf{v} = 2\mathbf{u}_1 + 1 \cdot \mathbf{u}_2 + 0 \cdot \mathbf{u}_3$.

(c) *Let* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}$ *be vectors in* $\mathbb{R}^3$, *given by* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}$.

We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Define $U = \left[\begin{array}{c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{array}\right]$.

The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is $C = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 3 & -2 & 1 & 7 \\ -1 & 3 & -5 & 3 \end{array}\right]$.

We find some row-echelon form and then the reduced row-echelon form C' which are row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots\cdots \longrightarrow C'$$

Note that $C^\sharp$ is a row-echelon form which is row-equivalent to C.

Note that the last column of $C^\sharp$ contains a leading one.

(So, without computing C' explicitly, we still know that the last column of C' is a pivot column.)

Then $\mathcal{LS}(U, \mathbf{v})$ with unknown $\mathbf{x}$ in $\mathbb{R}^3$ is inconsistent.

Therefore $\mathbf{v}$ is not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

(c) *Let* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}$ *be vectors in* $\mathbb{R}^3$, *given by* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}$.

We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Define $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\right]$.

The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is $C = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 3 & -2 & 1 & 7 \\ -1 & 3 & -5 & 3 \end{array}\right]$.

We find some row-echelon form and then the reduced row-echelon form C' which are row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^{\sharp} = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots\cdots \longrightarrow C'$$

There is some leading one in the last column.

Note that $C^{\sharp}$ is a row-echelon form which is row-equivalent to C.

Note that the last column of $C^{\sharp}$ contains a leading one.

(So, without computing C' explicitly, we still know that the last column of C' is a pivot column.)

Then $\mathcal{LS}(U, \mathbf{v})$ with unknown $\mathbf{x}$ in $\mathbb{R}^3$ is inconsistent.

Therefore $\mathbf{v}$ is not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

(d) *Let* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{v}$ *be vectors in* $\mathbb{R}^4$, *given by*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 9 \\ 0 \end{bmatrix}.$$

We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.

Define $U = \left[\begin{array}{c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{array}\right]$. The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is $C = \left[\begin{array}{cccc|c} 1 & 2 & 7 & 1 & -1 \\ 1 & 1 & 3 & 1 & 0 \\ 3 & 2 & 5 & -1 & 9 \\ 1 & -1 & -5 & 2 & 0 \end{array}\right]$.

We find some row-echelon form and then the reduced row-echelon form C' which are row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C' = \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which is the augmented matrix representation of the system $\left\{\begin{array}{rcrcrcrcr} x_1 & & & - & x_3 & & & = & 3 \\ & & x_2 & + & 4x_3 & & & = & -1 \\ & & & & & & x_4 & = & -2 \\ & & & & & & 0 & = & 0 \end{array}\right.$

Hence $\mathcal{LS}(U, \mathbf{v})$ with unknown $\mathbf{x}$ in $\mathbb{R}^4$ has some solution, for instance, $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ -2 \end{bmatrix}$.

Then $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$. In fact, $\mathbf{v} = 3\mathbf{u}_1 - 1 \cdot \mathbf{u}_2 + 0 \cdot \mathbf{u}_3 - 2\mathbf{u}_4$.

(d) *Let* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{v}$ *be vectors in* $\mathbb{R}^4$*, given by*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 9 \\ 0 \end{bmatrix}.$$

We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.

Define $U = \left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4\ \right]$. The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is $C = \left[\begin{array}{cccc|c} 1 & 2 & 7 & 1 & -1 \\ 1 & 1 & 3 & 1 & 0 \\ 3 & 2 & 5 & -1 & 9 \\ 1 & -1 & -5 & 2 & 0 \end{array}\right]$.

We find some row-echelon form and then the reduced row-echelon form C' which are row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C' = \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

There is no leading one in the last column.

which is the augmented matrix representation of the system $\left\{\begin{array}{rcrcrcrcr} x_1 & & & - & x_3 & & & = & 3 \\ & & x_2 & + & 4x_3 & & & = & -1 \\ & & & & & & x_4 & = & -2 \\ & & & & & & 0 & = & 0 \end{array}\right.$

Hence $\mathcal{LS}(U, \mathbf{v})$ with unknown $\mathbf{x}$ in $\mathbb{R}^4$ has some solution, for instance, $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ -2 \end{bmatrix}$.

Then $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$. In fact, $\mathbf{v} = 3\mathbf{u}_1 - 1 \cdot \mathbf{u}_2 + 0 \cdot \mathbf{u}_3 - 2\mathbf{u}_4$.

(e) *Let* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{v}$ *be vectors in* $\mathbb{R}^4$, *given by*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.

Define $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\middle|\, \mathbf{u}_4 \,\right]$.

The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is $C = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 3 & 4 & 4 & 3 & 1 \\ 2 & 2 & 1 & 1 & 0 \end{array}\right]$.

We find some row-echelon form and then the reduced row-echelon form C' which are row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots\cdots \longrightarrow C'$$

Note that $C^\sharp$ is a row-echelon form which is row-equivalent to C.

Note that the last column of $C^\sharp$ contains a leading one.

(So, without computing C' explicitly, we still know that the last column of C' is a pivot column.)

Then $\mathcal{LS}(U, \mathbf{v})$ with unknown $\mathbf{x}$ in $\mathbb{R}^4$ is inconsistent.

Therefore $\mathbf{v}$ is not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.

(e) *Let* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{v}$ *be vectors in* $\mathbb{R}^4$, *given by*

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.

Define $U = \left[\, \mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \mathbf{u}_3 \,|\, \mathbf{u}_4 \,\right]$.

The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is $C = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 3 & 4 & 4 & 3 & 1 \\ 2 & 2 & 1 & 1 & 0 \end{array}\right]$.

We find some row-echelon form and then the reduced row-echelon form C' which are row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C^\sharp = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots\cdots\cdots \longrightarrow C'$$

There is some leading one in the last column.

Note that $C^\sharp$ is a row-echelon form which is row-equivalent to C.
Note that the last column of $C^\sharp$ contains a leading one.
(So, without computing C' explicitly, we still know that the last column of C' is a pivot column.)
Then $\mathcal{LS}(U, \mathbf{v})$ with unknown $\mathbf{x}$ in $\mathbb{R}^4$ is inconsistent.
Therefore $\mathbf{v}$ is not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.

(f) *Let* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6, \mathbf{v}$ *be vectors in* $\mathbb{R}^4$, *given by*

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ -4 \\ -6 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 3 \\ -1 \\ 3 \\ 5 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \mathbf{u}_6 = \begin{bmatrix} -7 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 12 \\ 0 \\ 5 \\ 10 \end{bmatrix}.$$

We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$.

Define $U = \left[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5 \mid \mathbf{u}_6\ \right]$.

The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is $C = \left[\begin{array}{cccccc|c} 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ -1 & 2 & 1 & -1 & 0 & -2 & 0 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{array}\right]$.

We find some row-echelon form and then the reduced row-echelon form C' which are row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C' = \left[\begin{array}{cccccc|c} 1 & -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which is the augmented matrix representation of the system $\left\{\begin{array}{rcrcrcrcrcrcrcl} x_1 & - & 2x_2 & & & & & & & + & x_6 & = & 1 \\ & & & & x_3 & & & + & x_5 & - & 2x_6 & = & 3 \\ & & & & & & x_4 & + & x_5 & - & x_6 & = & 2 \\ & & & & & & & & & & 0 & = & 0 \end{array}\right.$.

Hence $\mathcal{LS}(U, \mathbf{v})$ with unknown $\mathbf{x}$ in $\mathbb{R}^6$ has some solution, for instance, $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

Then $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$. In fact, $\mathbf{v} = 1 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + 3\mathbf{u}_3 + 2\mathbf{u}_4 + 0 \cdot \mathbf{u}_5 + 0 \cdot \mathbf{u}_6$.

(f) *Let* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6, \mathbf{v}$ *be vectors in* $\mathbb{R}^4$, *given by*

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ -4 \\ -6 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 3 \\ -1 \\ 3 \\ 5 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \mathbf{u}_6 = \begin{bmatrix} -7 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 12 \\ 0 \\ 5 \\ 10 \end{bmatrix}.$$

We want to determine whether $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$.

Define $U = \left[\, \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5 \mid \mathbf{u}_6 \,\right]$.

The augmented matrix representation of $\mathcal{LS}(U, \mathbf{v})$ is $C = \left[\begin{array}{cccccc|c} 0 & 0 & 2 & 3 & 5 & -7 & 12 \\ -1 & 2 & 1 & -1 & 0 & -2 & 0 \\ 2 & -4 & -1 & 3 & 2 & 1 & 5 \\ 3 & -6 & -1 & 5 & 4 & 0 & 10 \end{array}\right]$.

We find some row-echelon form and then the reduced row-echelon form C' which are row-equivalent to C:

$$C \longrightarrow \cdots\cdots\cdots \longrightarrow C' = \left[\begin{array}{cccccc|c} 1 & -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

There is no leading one in the last column.

which is the augmented matrix representation of the system $\left\{ \begin{array}{rcrcrcrcrcrcl} x_1 & - & 2x_2 & & & & & & & + & x_6 & = & 1 \\ & & & & x_3 & & & + & x_5 & - & 2x_6 & = & 3 \\ & & & & & & x_4 & + & x_5 & - & x_6 & = & 2 \\ & & & & & & & & & & 0 & = & 0 \end{array} \right.$.

Hence $\mathcal{LS}(U, \mathbf{v})$ with unknown $\mathbf{x}$ in $\mathbb{R}^6$ has some solution, for instance, $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

Then $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6$. In fact, $\mathbf{v} = 1 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + 3\mathbf{u}_3 + 2\mathbf{u}_4 + 0 \cdot \mathbf{u}_5 + 0 \cdot \mathbf{u}_6$.

7. Recall the definition for the notion of *consistent systems.*

Let A be an $(m \times n)$-matrix and $\mathbf{b}$ be a vector in $\mathbb{R}^m$.
The system $\mathcal{LS}(A, \mathbf{b})$ is said to be consistent if and only if

$$\mathcal{LS}(A, \mathbf{b}) \text{ has at least one solution in } \mathbb{R}^n.$$

In the light of this, Theorem (1) is an immediate consequence of Theorem (F):

Theorem (1).
Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^m$, and U be the $(m \times n)$-matrix given by $U = [\, \mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \cdots \,|\, \mathbf{u}_n \,]$.

The statements below are logically equivalent:

(a) *Every vector in $\mathbb{R}^m$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.*

(b) *For any $\mathbf{v} \in \mathbb{R}^m$, the system $\mathcal{LS}(U, \mathbf{v})$ has at least one solution in $\mathbb{R}^n$.*

(c) *The reduced row-echelon form which is row-equivalent to U is of rank m.*

Remark.
Statement (c) is just another way of stating that for each $\mathbf{v} \in \mathbb{R}^m$, the reduced row-echelon form which is row-equivalent to the augmented matrix representation of the system $\mathcal{LS}(U, \mathbf{v})$ will have m pivot columns, all within the first n columns.

Theorem (2) below is a consequence of Theorem (E) in the Handout *Existence and uniqueness of solutions for a system of linear equations whose coefficient matrix is a square matrix* :

Theorem (2).
Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are vectors in* $\mathbb{R}^n$.
Define the $(n \times n)$*-square matrix* U *by* $U = [\,\mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \cdots \,|\, \mathbf{u}_n\,]$.
Then the statements below are logically equivalent:

(a) *Every vector in* $\mathbb{R}^n$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

(b) U *is non-singular.*

(c) U *is invertible.*

Remark.
This result will be merged with Theorem (E) in the Handout *Existence and uniqueness of solutions for a system of linear equations whose coefficient matrix is a square matrix* later, alongside more re-formulations for the notion of *non-singularity*.