0. Recall the properties of the notion of *null space of a given matrix* stated in the theoretical result below:

Let A be an $(m \times n)$-matrix. The statements below hold:

(1) $\mathbf{0} \in \mathcal{N}(A)$.

(2) *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, *if* $\mathbf{u}, \mathbf{v} \in \mathcal{N}(A)$ *then* $\mathbf{u} + \mathbf{v} \in \mathcal{N}(A)$.

(3) *For any* $\mathbf{u} \in \mathbb{R}^n$, *for any* $\alpha \in \mathbb{R}$, *if* $\mathbf{u} \in \mathcal{N}(A)$ *then* $\alpha\mathbf{u} \in \mathcal{N}(A)$.

(4) *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, *for any* $\alpha, \beta \in \mathbb{R}$, *if* $\mathbf{u}, \mathbf{v} \in \mathcal{N}(A)$ *then* $\alpha\mathbf{u} + \beta\mathbf{v} \in \mathcal{N}(A)$.

Also recall the properties of the notion of *column space of a given matrix* stated in the theoretical result below:

Suppose H is a $(p \times q)$-matrix. Then the statements below hold.

(1) $\mathbf{0}_p \in \mathcal{C}(H)$.

(2) *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$, *if* $\mathbf{x} \in \mathcal{C}(H)$ *and* $\mathbf{y} \in \mathcal{C}(H)$ *then* $\mathbf{x} + \mathbf{y} \in \mathcal{C}(H)$.

(3) *For any* $\mathbf{x} \in \mathbb{R}^p$, *for any* $\alpha \in \mathbb{R}$, *if* $\mathbf{x} \in \mathcal{C}(H)$ *then* $\alpha\mathbf{x} \in \mathcal{C}(H)$.

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They motivate the definition for the notion of *subspaces of* $\mathbb{R}^n$.

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Suppose H is a $(p \times q)$-matrix. Then the statements below hold.

(1) $\mathbf{0}_p \in \mathcal{C}(H)$.

(2) *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$, *if* $\mathbf{x} \in \mathcal{C}(H)$ *and* $\mathbf{y} \in \mathcal{C}(H)$ *then* $\mathbf{x} + \mathbf{y} \in \mathcal{C}(H)$.

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Statements (1), (2), (3), (4) here resemble the respective statements above: just change 'n' to 'p', and '$\mathcal{N}(A)$' to '$\mathcal{C}(H)$'.

This suggests the presence of some deeper mathematical structure behind 'null space' and 'column space'.

They motivate the definition for the notion of *subspaces of* $\mathbb{R}^n$.

1. **Definition. (Subspaces of $\mathbb{R}^n$.)**

Let W be a set of vectors in $\mathbb{R}^n$.

W is said to constitute a subspace of $\mathbb{R}^n$ if and only if the statements (S1), (S2), (S3) *hold:*

(S1) $\mathbf{0}_n \in W$.

(S2) *For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, if $\mathbf{u} \in W$ and $\mathbf{v} \in W$ then $\mathbf{u} + \mathbf{v} \in W$.*

(S3) *For any vector $\mathbf{u} \in \mathbb{R}^n$, for any $\alpha \in \mathbb{R}$, if $\mathbf{u} \in W$ then $\alpha\mathbf{u} \in W$.*

Remarks.

(a) Some people reads (S2) as:

‘*vector addition is closed in W.*’

(b) Some people reads (S3) as:

‘*scalar multiplication is closed in W.*’

(c) Some people like to combine (S2) and (S3) into

($\star$) ‘*For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, for any $\alpha \in \mathbb{R}$, if $\mathbf{u} \in W$ and $\mathbf{v} \in W$ then $\alpha\mathbf{u} + \mathbf{v} \in W$.*’

(d) We have already learnt plenty of examples of this concept without knowing that they are examples.

2. **Examples on subspaces.**

(a) The null space of every $(m \times n)$-matrix is a subspace of $\mathbb{R}^n$.

(b) The column space of every $(p \times q)$-matrix is a subspace of $\mathbb{R}^p$.

(c) Suppose $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$ are vectors in $\mathbb{R}^m$. Then the span of $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$ is subspace of $\mathbb{R}^m$.

(d) ('Extreme case'.)
$\{\mathbf{0}_n\}$ is a subspace of $\mathbb{R}^n$. It is called the zero subspace of $\mathbb{R}^n$.

(In fact, $\mathcal{N}(I_n) = \{\mathbf{0}_n\}$.)

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$\mathbb{R}^n$ is a subspace of $\mathcal{N}(\mathcal{O}_{n \times n}) = \mathbb{R}^n$.

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Reason. These statements hold: (1) $0_m \in \mathrm{Span}(\{z_1, z_2, \ldots, z_n\})$.
(2) For any $x, y \in \mathbb{R}^m$, if $x, y \in \mathrm{Span}(\{z_1, z_2, \ldots, z_n\})$ then $x+y \in \mathrm{Span}(\{z_1, z_2, \ldots, z_n\})$.
(3) For any $x \in \mathbb{R}^m$, for any $\alpha \in \mathbb{R}$, if $x \in \mathrm{Span}(\{z_1, z_2, \ldots, z_n\})$ then $\alpha x \in \mathrm{Span}(\{z_1, z_2, \ldots, z_n\})$.

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$\mathbb{R}^n$ is a subspace of $\mathcal{N}(\mathcal{O}_{n\times n}) = \mathbb{R}^n$.

3. **Lemma (1).**

Suppose W is a subspace of $\mathbb{R}^n$.

Then for any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, for any $\alpha, \beta \in \mathbb{R}$, if $\mathbf{u}, \mathbf{v} \in W$ then $\alpha\mathbf{u} + \beta\mathbf{v} \in W$.

Proof of Lemma (1).

Suppose W is a subspace of $\mathbb{R}^n$.

Pick any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Pick any $\alpha, \beta \in \mathbb{R}$.

Suppose $\mathbf{u}, \mathbf{v} \in W$.

Then by (S3), $\alpha\mathbf{u}, \beta\mathbf{v} \in W$.

Therefore by (S2), $\alpha\mathbf{u} + \beta\mathbf{v} \in W$.

Remarks.

(a) It follows that $-\mathbf{u}$, $\mathbf{v} - \mathbf{u}$ belong to W whenever $\mathbf{u}, \mathbf{v}$ belongs to W.

(b) So a subspace of $\mathbb{R}^n$ is very much like a copy of $\mathbb{R}^k$ (for some k) 'sitting inside $\mathbb{R}^n$' and containing the zero vector in $\mathbb{R}^n$.
We may perform addition and scalar multiplication on the vectors in this 'copy of $\mathbb{R}^k$' in an arbitrary manner without the resultants 'leaving' it.

3. **Lemma (1).**

Suppose W is a subspace of $\mathbb{R}^n$.

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Suppose W is a subspace of $\mathbb{R}^n$.

Pick any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Pick any $\alpha, \beta \in \mathbb{R}$.

Suppose $\mathbf{u}, \mathbf{v} \in W$.

Then by (S3), $\alpha\mathbf{u}, \beta\mathbf{v} \in W$.

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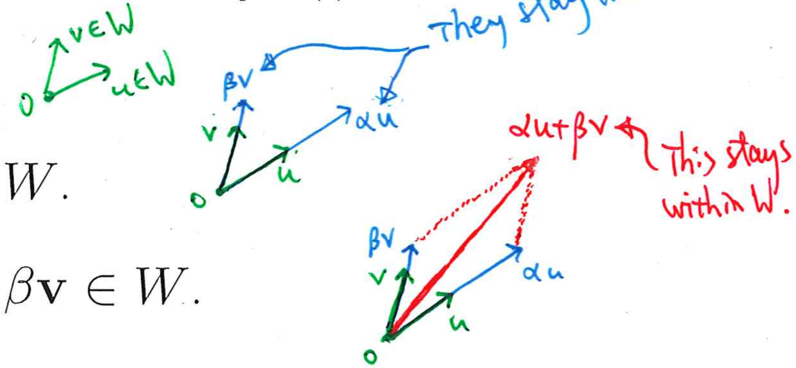

Reminder.
W is a subspace of $\mathbb{R}^n$ exactly when (S1), (S2), (S3) hold:
(S1) $0_n \in W$
(S2) For any $u, v \in \mathbb{R}^n$ if $u, v \in W$ then $u+v \in W$.
(S3) For any $u \in \mathbb{R}^n$, for any $\alpha \in \mathbb{R}$, if $u \in W$ then $\alpha u \in W$.

Remarks.

This is $(-1)u$. This is $(-1)u + 1v$.

(a) It follows that $-\mathbf{u}$, $\mathbf{v} - \mathbf{u}$ belong to W whenever $\mathbf{u}, \mathbf{v}$ belongs to W.

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We may perform addition and scalar multiplication on the vectors in this 'copy of $\mathbb{R}^k$' in an arbitrary manner without the resultants 'leaving' it.

4. **Lemma (2).**

Suppose W is a subspace of $\mathbb{R}^n$.

Then every linear combination of vectors in W belongs to W.

Proof of Lemma (2).

Suppose W is a subspace of $\mathbb{R}^n$.

Suppose $\mathbf{x}$ is a linear combination of vectors in W.

By definition, there are some $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in W$ and $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{R}$ so that

$$\mathbf{x} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k.$$

By Lemma (1), $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 \in W$.

By Lemma (1) again, $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 = (\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2) + \alpha_3\mathbf{u}_3 \in W$.

By Lemma (1) again, $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \alpha_4\mathbf{u}_4 = (\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3) + \alpha_4\mathbf{u}_4 \in W$.

Repeatedly by Lemma (1), we deduce that $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k \in W$. The result follows.

Proof of Lemma (2), formally presented in the form of mathematical induction.

Suppose W is a subspace of $\mathbb{R}^n$. We are going to verify the statement
'For any positive integer s, if $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_s \in W$ and $\alpha_1, \alpha_2, \cdots, \alpha_s \in \mathbb{R}$ then $\alpha_1\mathbf{u_1}+\alpha_2\mathbf{u}_2+\cdots+\alpha_s\mathbf{u}_s \in W$.'
Denote by $P(s)$ the proposition below:

'If $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_s \in W$ and $\alpha_1, \alpha_2, \cdots, \alpha_s \in \mathbb{R}$ then $\alpha_1\mathbf{u_1} + \alpha_2\mathbf{u}_2 + \cdots + \alpha_s\mathbf{u}_s \in W$.'

$P(1)$ follows from (S3) immediately.

Suppose $P(k)$ is true.

Note that $P(k+1)$ reads:
'If $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k, \mathbf{u}_{k+1} \in W$ and $\alpha_1, \alpha_2, \cdots, \alpha_k, \alpha_{k+1} \in \mathbb{R}$ then $\alpha_1\mathbf{u_1}+\alpha_2\mathbf{u}_2+\cdots+\alpha_k\mathbf{u}_k+\alpha_{k+1}\mathbf{u}_{k+1} \in W$.'
With the help of $P(k)$, we verify $P(k+1)$:

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k, \mathbf{u}_{k+1} \in W$ and $\alpha_1, \alpha_2, \cdots, \alpha_k, \alpha_{k+1} \in \mathbb{R}$.
By $P(k)$, $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k \in W$.
By (S3), $\alpha_{k+1}\mathbf{u}_{k+1} \in W$.
Then, by (S2), $\alpha_1\mathbf{u_1} + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k + \alpha_{k+1}\mathbf{u}_{k+1} \in W$.
Therefore $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(s)$ is true for any positive integer s.

5. **Corollary to Lemma (2).**

The statements below hold:

(a) *Suppose A is an $(m \times n)$-matrix.*
Then every linear combination of vectors in $\mathcal{N}(A)$ is a vector in $\mathcal{N}(A)$.

(b) *Suppose H is a $(p \times q)$-matrix.*
Then every linear combination of vectors in $\mathcal{C}(H)$ is a vector in $\mathcal{C}(H)$.

(c) *Suppose $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$ are vectors in $\mathbb{R}^m$.*
Then every linear combination of vectors in $\mathsf{Span}\,(\{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n\})$ is a vector in $\mathsf{Span}\,(\{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n\})$.

6. **Comments.**

It will turn out that:

- every subspace of $\mathbb{R}^n$ is the null space of some matrix with n columns,
- it is the column space of some matrix with n rows, and
- it is the span of some collection of vectors in $\mathbb{R}^n$.

You may wonder why we still bother to introduce the notion of *subspace*, when this concept apparently yields 'nothing' we don't know already from the study of other concepts.

In fact, the power of *algebra* (of which *linear algebra* is a part) is in

the unifying of (seemingly unrelated) concepts.

Lemma (2) and Corollary to Lemma (2) is a case in point.

Having proved Lemma (2), which applies to arbitrary subspaces of $\mathbb{R}^n$, there will be no need to verify the three statements in Corollary to Lemma (2) separately (and repeat the same arguments for Lemma (2) in the respective separate justifications for the three statements in Corollary to Lemma (2)).

This will save a lot of time and effort, (which can be put to better use elsewhere).

7. **Lemma (3).**

Let W be a set of vector in $\mathbb{R}^n$. Suppose W is non-empty as a set.

Further suppose that every linear combination of vectors in W belongs to W.

Then W is a subspace of $\mathbb{R}^n$.

Proof of Lemma (3).

Let W be a set of vector in $\mathbb{R}^n$. Suppose W is non-empty as a set.

Further suppose that every linear combination of vectors in W belongs to W.

- Because W is non-empty, we may pick some $\mathbf{z} \in W$.
  Note that $\mathbf{0}_n = \mathbf{z} - \mathbf{z}$, and $\mathbf{z} - \mathbf{z}$ is a linear combination of $\mathbf{z}, \mathbf{z}$.
  Then by assumption, $\mathbf{0}_n$ belongs to W.
  [Hence (S1) is satisfied.]
- Pick any $\mathbf{u}, \mathbf{v} \in W$. By definition, $\mathbf{u} + \mathbf{v}$ is a linear combination of $\mathbf{u}, \mathbf{v}$.
  Then $\mathbf{u} + \mathbf{v}$ belongs to W.
  [Hence W satisfies (S2).]
- Pick any $\mathbf{u} \in W$. Pick any $\alpha \in \mathbb{R}$. By definition, $\alpha\mathbf{u}$ is a linear combination of $\mathbf{u}$.
  Then $\alpha\mathbf{u}$ belongs to W.
  [Hence W satisfies (S3).]

It follows that W is a subspace of $\mathbb{R}^n$.

8. Combining Lemma (2), Lemma (3), we obtain Theorem (E) below.

Theorem (E). (Characterization of subspaces of $\mathbb{R}^n$ as special types of subsets of $\mathbb{R}^n$.)

Let W be a set of vectors in $\mathbb{R}^n$. Suppose W is a non-empty set of vectors. Then

$$W \text{ is a subspace of } \mathbb{R}^n$$

if and only if

$$\text{every linear combination of vectors in } W \text{ belongs to } W.$$

Remark.

So what is so special about subspaces of $\mathbb{R}^n$?

They are those and only those non-empty subsets of $\mathbb{R}^n$, for which it happens that there is definitely no chance for a linear combination of vectors in such a set to 'fall outside' the set.

9. **Further examples.**

(a) *Let* A *be an* $(m \times n)$*-matrix, and* Z *be a subspace of* $\mathbb{R}^m$.
Define $W = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \in Z\}$.
Then W *is a subspace of* $\mathbb{R}^n$.

Remark.

When $Z = \{\mathbf{0}_m\}$, $W = \mathcal{N}(A)$. So we may think of this example as a generalization of the notion of null space.

(b) *Let* H *be an* $(p \times q)$*-matrix, and* Z *be a subspace of* $\mathbb{R}^q$.
Define

$$W = \left\{ \mathbf{y} \in \mathbb{R}^p : \begin{array}{l} \textit{There exist some } \mathbf{u} \in Z \\ \textit{such that } \mathbf{y} = H\mathbf{u} \end{array} \right\}.$$

Then W *is a subspace of* $\mathbb{R}^p$.

Remark.

When $Z = \mathbb{R}^q$, $W = \mathcal{C}(A)$. So we may think of this example as a generalization of the notion of column space.

(c) *Let V, W be subspaces of $\mathbb{R}^n$.*
Recall that the intersection of V, W is the set

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in V \text{ and } \mathbf{x} \in W\}.$$

It is denoted by $V \cap W$.
$V \cap W$ is a subspace of $\mathbb{R}^n$.

(d) *Let V, W be subspaces of $\mathbb{R}^n$.*
Define the set $V + W$, which is called the sum of V, W, by

$$V + W = \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{array}{l} \text{There exist some } \mathbf{s} \in V,\ \mathbf{t} \in W \\ \text{such that } \mathbf{x} = \mathbf{s} + \mathbf{t} \end{array} \right\}.$$

Then $V + W$ is a subspace of $\mathbb{R}^n$.
Remark.
$V + W$ is the collection of those and only those vectors in $\mathbb{R}^n$ which can be expressed as a sum of two vectors, one in V and the other in W.

(e) *Let S be a set of vectors of $\mathbb{R}^n$.*
Define the set $S^\perp$, which is called the perp of S, by

$$S^\perp = \{\mathbf{x} \in \mathbb{R}^n : \text{ For any } \mathbf{u} \in S,\ \mathbf{u}^t\mathbf{x} = 0.\ \}.$$

Then $S^\perp$ is a subspace of $\mathbb{R}^n$.