1. **Definition. (Linear Combination.)**

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.

   Let $\mathbf{v}$ *be a vector in* $\mathbb{R}^m$.

   We say $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *if the statement* (†) *holds:*

   (†) *There exist some real numbers* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *such that* $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n$.

   The expression $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n$ *on its own is called the linear combination of the vectors* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *with respect to the scalars* $\alpha_1, \alpha_2, \cdots, \alpha_n$.

2. **Lemma (A). ('Dictionary' between linear combinations and matrix-vector products.)**

   Let A *be an* $(m \times n)$*-matrix, and* $\mathbf{t}$ *be a vector in* $\mathbb{R}^n$.

   Suppose that for each $j = 1, 2, \cdots, n$, *the* j*-th column of* A *is* $\mathbf{a}_j$ *and the* j*-th entry of* $\mathbf{t}$ *is* t_j. *(So* $A = [\ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_n\ ]$ *and* $\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$*.)*

   Then $A\mathbf{t} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n$.

3. **Proof of Lemma (A).**

   For each i, j, we denote the (i,j)-th entry of A by a_{ij}.

   - The i-th entry of $A\mathbf{t}$ is given by $\sum_{j=1}^{n} a_{ij}t_j = t_1a_{i1} + t_2a_{i2} + \cdots + t_na_{in}$.
   - For each j, the i-th entry of $\mathbf{a}_j$ (which is the j-th column of A) is a_{ij}.
     Then the i-th entry of $t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n$ is $t_1a_{i1} + t_2a_{i2} + \cdots + t_na_{in}$.

   The corresponding entries of $A\mathbf{t}$, $t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n$ agree with each other.

   Hence $A\mathbf{t} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n$ indeed.

   Remark. Lemma (A) looks innocent, but it will serve as a useful tool in various situations.

4. **Simple concrete examples.**

   (a) $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

   So $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ is the linear combination of $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ with respect to the scalars $1, 2, 3, 4, 5$.

   A manifestation of the same relation is the equality below:

   $$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \left[\begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}.$$

   (b) $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 6 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$.

   So $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ is also a linear combination of $\begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$ with respect to the scalars $1, -1, 2, -1, 0$.

   A manifestation of the same relation is the equality below:

   $$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \left[\begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 \\ 5 & 4 & 1 & 0 & 0 \\ 7 & 6 & 3 & 3 & 0 \\ 9 & 8 & 5 & 6 & 2 \end{array}\right] \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \\ 0 \end{bmatrix}.$$

5. **Theorem (1).**

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.

   The statements below are true:

   (a) *The zero vector* $\mathbf{0}$ *in* $\mathbb{R}^m$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

   (b) *The sum of any two linear combinations of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

   (c) *Every scalar multiple of any linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

6. **Proof of Theorem (1).**

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^m$.

   (a) [Ask: Can we name some appropriate real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ for which the equality $\mathbf{0} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n$ holds?]
   We have $\mathbf{0} = 0 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + \cdots + 0 \cdot \mathbf{u}_n$.
   Then by definition, $\mathbf{0}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

   (b) Suppose $\mathbf{v}, \mathbf{w}$ are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.
   Then, by definition, there exist some real numbers $\beta_1, \beta_2, \cdots, \beta_n$ such that $\mathbf{v} = \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \cdots + \beta_n\mathbf{u}_n$.
   Also, there exist some real numbers $\gamma_1, \gamma_2, \cdots, \gamma_n$ such that $\mathbf{w} = \gamma_1\mathbf{u}_1 + \gamma_2\mathbf{u}_2 + \cdots + \gamma_n\mathbf{u}_n$.
   [Ask: Can we name some appropriate real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ for which the equality $\mathbf{v} + \mathbf{w} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n$ holds?]
   Note that $\mathbf{v} + \mathbf{w} = ... = (\beta_1 + \gamma_1)\mathbf{u}_1 + (\beta_2 + \gamma_2)\mathbf{u}_2 + \cdots + (\beta_n + \gamma_n)\mathbf{u}_n$, and $\beta_1 + \gamma_1, \beta_2 + \gamma_2, \cdots, \beta_n + \gamma_n$ are real numbers.
   Then by definition, $\mathbf{v} + \mathbf{w}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

   (c) Exercise.

7. **Theorem (B).**

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.

   Every linear combination of (finitely many) linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

   Remark. In fact, Theorem (B) is saying the same thing as Statement (b) and Statement (c) in Theorem (1) combined.

   Its conclusion part can be formulated as:

   > *For any* $\mathbf{x} \in \mathbb{R}^m$*, if* $\mathbf{x}$ *is a linear combination of some* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p \in \mathbb{R}^m$ *which are themselves linear combinations of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$*, then* $\mathbf{x}$ *itself is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

8. **Proof of Theorem (B).**

   [This argument carries the same essence of the argument for Statement (b) and Statement (c) in Theorem (1).]

   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^m$.

   Pick any $\mathbf{x} \in \mathbb{R}^m$.

   Suppose $\mathbf{x}$ is a linear combination of (finitely many) vectors in $\mathbb{R}^m$, say, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$, which are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

   [Reminder: We want to see why $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.]

   By definition, $\mathbf{x}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$.

   Then there exist some $\alpha_1, \alpha_2, \cdots, \alpha_p \in \mathbb{R}$ such that $\mathbf{x} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p$.

   [Ask: Can we link up the $\mathbf{u}_j$'s with the $\mathbf{v}_i$'s so as to see that $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$?]

   By assumption, for each $j = 1, 2, \cdots, p$, there exist some $\beta_{1j}, \beta_{2j}, \cdots, \beta_{nj} \in \mathbb{R}$ such that $\mathbf{v}_j = \beta_{1j}\mathbf{u}_1 + \beta_{2j}\mathbf{u}_2 + \cdots + \beta_{nj}\mathbf{u}_n$.

   Then

$$\begin{aligned}
\mathbf{x} &= \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p \\
&= \alpha_1(\beta_{11}\mathbf{u}_1 + \beta_{21}\mathbf{u}_2 + \cdots + \beta_{n1}\mathbf{u}_n) + \alpha_2(\beta_{12}\mathbf{u}_1 + \beta_{22}\mathbf{u}_2 + \cdots + \beta_{n2}\mathbf{u}_n) \\
&\quad + \cdots + \alpha_p(\beta_{1p}\mathbf{u}_1 + \beta_{2p}\mathbf{u}_2 + \cdots + \beta_{np}\mathbf{u}_n) \\
&= (\beta_{11}\alpha_1 + \beta_{12}\alpha_2 + \cdots + \beta_{1p}\alpha_p)\mathbf{u}_1 + (\beta_{21}\alpha_1 + \beta_{22}\alpha_2 + \cdots + \beta_{2p}\alpha_p)\mathbf{u}_2 \\
&\quad + \cdots + (\beta_{n1}\alpha_1 + \beta_{n2}\alpha_2 + \cdots + \beta_{np}\alpha_p)\mathbf{u}_n
\end{aligned}$$

   Note $(\beta_{k1}\alpha_1 + \beta_{k2}\alpha_2 + \cdots + \beta_{kp}\alpha_p)$ is a real number for each $k = 1, 2, \cdots, n$.

   Then $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

9. **Alternative argument for Theorem (B).**

By applying mathematical induction, and by consciously applying Theorem (1), we are going to verify the statement

> '*For any positive integer s, if $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_s$ are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and $\alpha_1, \alpha_2, \cdots, \alpha_s$ are real numbers then $\alpha_1\mathbf{v_1} + \alpha_2\mathbf{v}_2 + \cdots + \alpha_s\mathbf{v}_s$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.*'

Denote by $P(s)$ the proposition below:

> 'If $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_s$ are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and $\alpha_1, \alpha_2, \cdots, \alpha_s$ are real numbers then $\alpha_1\mathbf{v_1} + \alpha_2\mathbf{v}_2 + \cdots + \alpha_s\mathbf{v}_s$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.'

We verify $P(1)$:

> Suppose $\mathbf{v}_1$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and α_1 is a real number.
> Then by Theorem (1), $\alpha_1\mathbf{v}_1$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

Suppose $P(k)$ is true.

Note that $P(k+1)$ reads:

> 'If $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and $\alpha_1, \alpha_2, \cdots, \alpha_k, \alpha_{k+1}$ are real numbers then $\alpha_1\mathbf{v_1} + \alpha_2\mathbf{v}_2 + \cdots + \alpha_{k+1}\mathbf{v}_{k+1}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.'

With the help of $P(k)$, we verify $P(k+1)$:

> Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and $\alpha_1, \alpha_2, \cdots, \alpha_k, \alpha_{k+1}$ are real numbers.
> By $P(k)$, $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.
> By $P(1)$, $\alpha_{k+1}\mathbf{v}_{k+1}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.
> Then, by Theorem (1), $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k + \alpha_{k+1}\mathbf{v}_{k+1}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.
> Therefore $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(s)$ is true for any positive integer s.

10. We now state a pair of results (Lemma (2), Lemma (3)) describing whether square-matrix multiplication from the left to vectors 'preserves' linear relations amongst vectors.

Lemma (2).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ be vectors in $\mathbb{R}^m$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be real numbers.

Suppose A is an $(m \times m)$-square matrix, and $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

Then $A\mathbf{v}$ is a linear combination of the vectors $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

Lemma (3). (A 'partial converse' of Lemma (2).)

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ be vectors in $\mathbb{R}^m$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be real numbers.

Suppose A is a non-singular $(m \times m)$-square matrix, and $A\mathbf{v}$ is a linear combination of the vectors $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

Then $\mathbf{v}$ is a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

11. We combine Lemma (2) and Lemma (3) to obtain Theorem (C) below:

Theorem (C).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ be vectors in $\mathbb{R}^m$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be real numbers.

Suppose A is a non-singular $(m \times m)$-square matrix. Then the statements below are logically equivalent:

(a) *$\mathbf{v}$ is a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.*

(b) *$A\mathbf{v}$ is a linear combination of the vectors $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.*

Remark. In plain words, this result is saying that

> *linear relations amongst vectors (though not necessarily the individual vectors themselves) are preserved upon the multiplication by the same non-singular matrix from the left to the vectors.*

When we think in terms of row operations, this result is saying that

> *linear relations amongst vectors (though not necessarily the individual vectors themselves) are preserved upon the application of the same sequence of row operations to the vectors.*

12. **Proof of Lemma (2).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ be vectors in $\mathbb{R}^m$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be real numbers.

Suppose A is an $(m \times m)$-square matrix, and $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

By assumption, $\mathbf{v} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n$.

Then $A\mathbf{v} = A(\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n) = \alpha_1 A\mathbf{u}_1 + \alpha_2 A\mathbf{u}_2 + \cdots + \alpha_n A\mathbf{u}_n$.

Hence $A\mathbf{v}$ is a linear combination of $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

13. **Proof of Lemma (3).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ be vectors in $\mathbb{R}^m$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be real numbers.

Suppose A is a non-singular $(m \times m)$-square matrix, and $A\mathbf{v}$ is a linear combination of the vectors $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

By assumption, $A\mathbf{v} = \alpha_1 A\mathbf{u}_1 + \alpha_2 A\mathbf{u}_2 + \cdots + \alpha_n A\mathbf{u}_n$.

[Ask: Is $\mathbf{v}$ a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$?]

Since A is non-singular, A is invertible.

Therefore

$$\begin{aligned}
\mathbf{v} &= I_m \mathbf{v} = (A^{-1}A)\mathbf{v} = A^{-1}(A\mathbf{v}) \\
&= A^{-1}(\alpha_1 A\mathbf{u}_1 + \alpha_2 A\mathbf{u}_2 + \cdots + \alpha_n A\mathbf{u}_n) \\
&= \alpha_1 A^{-1}(A\mathbf{u}_1) + \alpha_2 A^{-1}(A\mathbf{u}_2) + \cdots + \alpha_n A^{-1}(A\mathbf{u}_n) \\
&= \alpha_1 (A^{-1}A)\mathbf{u}_1 + \alpha_2 (A^{-1}A)\mathbf{u}_2) + \cdots + \alpha_n (A^{-1}A)\mathbf{u}_n \\
&= \alpha_1 I_m \mathbf{u}_1 + \alpha_2 I_m \mathbf{u}_2 + \cdots + \alpha_n I_m \mathbf{u}_n \\
&= \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n
\end{aligned}$$

Then $\mathbf{v}$ is a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

14. **Theorem (4). (Generalization of Lemma (2) and Lemma (3).)**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ be vectors in $\mathbb{R}^m$. Let A be a $(p \times m)$-matrix.

(a) *Suppose $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.*
Then $A\mathbf{v}$ is a linear combination of the vectors $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$.

(b) *Suppose $\mathcal{N}(A) = \{\mathbf{0}\}$, and $A\mathbf{v}$ is a linear combination of the vectors $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$.*
Then $\mathbf{v}$ is a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

Proof of Theorem (4). Exercise.