

1. **Definition. (Invertibility.)**

Let A be an $(n \times n)$ -square matrix.

(a) *Suppose B is a $(n \times n)$ -square matrix.*

Further suppose $BA = I_n$ and $AB = I_n$.

Then we say B is a matrix inverse of A .

(b) *A is said to be invertible if and only if A has a matrix inverse.*

2. **Two (trivial) examples.**

(a) The identity matrix I_n is invertible, and a matrix inverse of it is I_n itself.

(b) The zero $(n \times n)$ -square matrix is not invertible.

3. **Lemma (α).** (**Uniqueness of matrix inverse.**)

Let A be an $(n \times n)$ -square matrix.

Suppose B, C are both matrix inverses of A .

Then $B = C$.

Proof of Lemma (α).

Under the assumption, we have $BA = I_n$ and $AC = I_n$.

Then

$$B = BI_n = B(AC) = (BA)C = I_n C = C.$$

Remarks.

- From now on there is no problem using the article *the* in writing the words *the matrix inverse of the invertible matrix blah-blah-blah*.
- For the same reason, it makes to label the matrix inverse of an invertible matrix, say, A , with something which involves the symbol ' A '.

From now on, we denote by A^{-1} the matrix inverse of such an invertible matrix A .

4. **Lemma (β). (Product of matrix inverses.)**

Let A, B be $(n \times n)$ -square matrices.

Suppose A, B are invertible.

Then the product AB is invertible with matrix inverse given by $(AB)^{-1} = B^{-1}A^{-1}$.

Proof of Lemma (β).

Under the assumption, we have $A^{-1}A = I_n$ and $AA^{-1} = I_n$.

Moreover, $B^{-1}B = I_n$ and $BB^{-1} = I_n$.

Write $C = AB$, and $D = B^{-1}A^{-1}$.

We have

$$DC = (B^{-1}A^{-1})(AB) = B^{-1}[A^{-1}(AB)] = B^{-1}[(A^{-1}A)B] = B^{-1}(I_n B) = B^{-1}B = I_n.$$

We also have

$$CD = (AB)(B^{-1}A^{-1}) = \dots\dots\dots = I_n.$$

Therefore, by the definition of matrix inverse and invertibility, C is invertible with matrix inverse D .

Then AB is invertible, and its matrix inverse is given by $(AB)^{-1} = B^{-1}A^{-1}$.

Remark.

By mathematical induction, we can prove this generalization of Lemma (β):

Let A_1, A_2, \dots, A_k be $(n \times n)$ -square matrices.

Suppose A_1, A_2, \dots, A_k are invertible.

Then the product $A_1 A_2 \cdots A_k$ is invertible with matrix inverse given by

$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}.$$

5. Corollary to Lemma (β).

Let A be an $(n \times n)$ -square matrix.

Suppose A is invertible.

Then, for each positive integer p , the matrix A^p is invertible with matrix inverse given by $(A^p)^{-1} = (A^{-1})^p$.

Proof. Exercise in mathematical induction.

6. Lemma (γ).

(a) *Every row operation matrix is invertible.*

Its matrix inverse is the row operation matrix corresponding to its reverse row operation.

(b) *Suppose H_1, H_2, \dots, H_k are row-operation matrices, and*

$$H = H_k \cdots H_2 H_1.$$

Then H is invertible, and its matrix inverse is given by

$$H^{-1} = H_1^{-1} H_2^{-1} \cdots H_k^{-1}.$$

Proof of Lemma (γ). [This is a straightforward calculation, though it requires patience.]

Importance of Item (a): Row-operation matrices constitute an important type of invertible matrices.

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Its matrix inverse is the row operation matrix corresponding to its reverse row operation.

(b) Suppose H_1, H_2, \dots, H_k are row-operation matrices, and

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Consequence of Item (a) and Lemma (β).

Proof of Lemma (γ).
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7. Lemma (δ). (Invertibility of matrix inverse.)

Let A be an $(n \times n)$ -square matrix.

Suppose A is invertible.

Then its matrix inverse A^{-1} is invertible, and the matrix inverse of A^{-1} is given by $(A^{-1})^{-1} = A$.

Proof of Lemma (δ).

Under the assumption, we have $A^{-1}A = I_n$ and $AA^{-1} = I_n$.

Write $B = A$, and $C = A^{-1}$.

Since $AA^{-1} = I_n$, we have $BC = I_n$.

Since $A^{-1}A = I_n$, we have $CB = I_n$.

Therefore $BC = I_n$ and $CB = I_n$.

Then, by the definition of matrix inverse and invertibility, C is invertible with matrix inverse B .

Therefore A^{-1} is invertible and its matrix inverse is given by $(A^{-1})^{-1} = A$.

8. **Lemma (ϵ).** (Invertibility implies non-singularity.)

Let A be an $(n \times n)$ -square matrix.

Suppose A invertible.

Then A is non-singular, and its matrix inverse A^{-1} is non-singular and invertible.

Proof of Lemma (ϵ).

Under the assumption, we have $A^{-1}A = I_n$ and $AA^{-1} = I_n$.

By Lemma (δ), A^{-1} is invertible.

Since $A^{-1}A = I_n$, we conclude from Lemma (2) that A is non-singular.

Since $AA^{-1} = I_n$, we conclude from Lemma (2) that A^{-1} is non-singular.

Remark. Recall Lemma (2):

Let C be a $(p \times p)$ -square matrix.

Suppose there exists some $(p \times p)$ -square matrix J such that $JC = I_p$.

Then C is non-singular.

9. **Lemma (ζ).** (Non-singularity implies invertibility.)

Let A be an $(n \times n)$ -square matrix.

Suppose A non-singular.

Then A is invertible, and its matrix inverse A^{-1} is non-singular and invertible.

Proof of Lemma (ζ).

Under the assumption, and according to Lemma (6), there exists some $(n \times n)$ -square matrix H such that H is non-singular, $HA = I_n$ and $AH = I_n$.

Now, by definition, A is invertible with its matrix inverse given by $A^{-1} = H$.

H is invertible by Lemma (δ).

10. **Theorem (B). (Equivalence of non-singularity and invertibility.)**

Let A be an $(n \times n)$ -square matrix.

A is non-singular if and only if A is invertible.

Furthermore, if A is invertible, then its matrix inverse A^{-1} is non-singular and invertible with matrix inverse given by $(A^{-1})^{-1} = A$.

11. **Corollary to Theorem (B).**

Let A be an $(n \times n)$ -square matrix.

The statements below are logically equivalent:

(a) *A is invertible.*

(b) *There exists some $(n \times n)$ -square matrix H such that $HA = I_n$.*

(c) *There exists some $(n \times n)$ -square matrix G such that $AG = I_n$.*

Proof of Corollary to Theorem (B).

- Suppose A is invertible.

Then A has a unique matrix inverse A^{-1} .

So it follows that $A^{-1}A = I_n$ and $AA^{-1} = I_n$.

- Suppose there exists some $(n \times n)$ -square matrix H such that $HA = I_n$.

Then, by Lemma (2), A is non-singular.

Therefore, by Theorem (B), A is invertible.

- Suppose there exists some $(n \times n)$ -square matrix G such that $AG = I_n$.

Then, by Lemma (2), G is non-singular.

Therefore, by Theorem (B), G is invertible.

We verify that $G^{-1} = A$:

We have $I_n = AG$.

Then $G^{-1} = I_n G^{-1} = (AG)G^{-1} = A(GG^{-1}) = AI_n = A$.

Then by Lemma (δ), A is invertible.

Remark.

With the help of Theorem (B) and its corollary, together with the calculations leading towards Lemma (6), we can ‘upgrade’ Theorem (A) to obtain Theorem (C).

12. **Theorem (C).** (Various re-formulations for the notions of non-singularity and invertibility.)

Let A be an $(n \times n)$ -square matrix. The statements below are logically equivalent:

- (a) A is non-singular.
- (b) For any vector \mathbf{v} in \mathbb{R}^n , if $A\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$.
- (c) The trivial solution is the only solution of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.
- (d) A is row-equivalent to I_n .
- (e) A is invertible.
- (f) There exists some $(n \times n)$ -square matrix H such that $HA = I_n$.
- (g) There exists some $(n \times n)$ -square matrix G such that $AG = I_n$.

Now suppose A is non-singular, with a sequence of row operations

$$A = C_1 \xrightarrow{\rho_1} C_2 \xrightarrow{\rho_2} \cdots \cdots \xrightarrow{\rho_{p-2}} C_{p-1} \xrightarrow{\rho_{p-1}} C_p = I_n,$$

and with H_k being the row-operation matrix corresponding to ρ_k for each k .

Then $[I_n|A^{-1}]$ is the resultant of the application of the same sequence of row operations $\rho_1, \rho_2, \dots, \rho_{p-1}$ starting from $[A|I_n]$:

$$[A|I_n] = [C_1|I_n] \xrightarrow{\rho_1} [C_2|H_1] \xrightarrow{\rho_2} [C_3|H_2H_1] \xrightarrow{\rho_3} \cdots \cdots \xrightarrow{\rho_{p-1}} [C_p|H_{p-1} \cdots H_2H_1] = [I_n|A^{-1}].$$

Moreover, A^{-1} and A are respectively given as products of row-operation matrices by

$$A^{-1} = H_{p-1} \cdots H_2H_1, \quad A = H_1^{-1}H_2^{-1} \cdots H_{p-1}^{-1}.$$

12. Theorem (C). (Various re-formulations for the notions of non-singularity and invertibility.)

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- (c) The trivial solution is the only solution of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.
- (d) A is row-equivalent to I_n .
- (e) A is invertible.
- (f) There exists some $(n \times n)$ -square matrix H such that $HA = I_n$.
- (g) There exists some $(n \times n)$ -square matrix G such that $AG = I_n$.

← Various ways for saying 'a given square matrix is non-singular.'

Now suppose A is non-singular, with a sequence of row operations

$$A = C_1 \xrightarrow{\rho_1} C_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{p-2}} C_{p-1} \xrightarrow{\rho_{p-1}} C_p = I_n,$$

△ This tells us how non-singular matrices are 'built up' as products of row-operation matrices.

and with H_k being the row-operation matrix corresponding to ρ_k for each k . ($C_{k+1} = H_k C_k$ for each k .)

Then $[I_n | A^{-1}]$ is the resultant of the application of the same sequence of row operations $\rho_1, \rho_2, \dots, \rho_{p-1}$ starting from $[A | I_n]$:

$$[A | I_n] = [C_1 | I_n] \xrightarrow{\rho_1} [C_2 | H_1] \xrightarrow{\rho_2} [C_3 | H_2 H_1] \xrightarrow{\rho_3} \cdots \xrightarrow{\rho_{p-1}} [C_p | H_{p-1} \cdots H_2 H_1] = [I_n | A^{-1}].$$

This suggests how to determine whether a square matrix is invertible, and to find its inverse.

Moreover, A^{-1} and A are respectively given as products of row-operation matrices by

$$A^{-1} = H_{p-1} \cdots H_2 H_1, \quad A = H_1^{-1} H_2^{-1} \cdots H_{p-1}^{-1}.$$

'Factorization' of both A and A^{-1} into products of row-operation matrices.

13. **Corollary to Theorem (C).**

The statements below hold:

- (a) *The matrix inverse of every invertible matrix is a product of finitely many row-operation matrices.*
- (b) *Every non-singular matrix is a product of finitely many row-operation matrices.*

14. Recall what Lemma (β) says (when put in plain words): the product of any two invertible matrices is an invertible. We now upgrade Lemma (β) with the help of Theorem (B), to obtain Lemma (β').

Lemma (β').

Let A, B be $(n \times n)$ -square matrices.

Suppose A, B are non-singular and invertible.

Then the product AB is non-singular and invertible.

15. The converse of Lemma (β'), as formulated below, is also true.

Lemma (η).

Let A, B be $(n \times n)$ -square matrices.

Suppose the product AB is non-singular and invertible.

Then each of A, B is non-singular and invertible.

13. Corollary to Theorem (C).

The statements below hold:

- (a) *The matrix inverse of every invertible matrix is a product of finitely many row-operation matrices.*
- (b) *Every non-singular matrix is a product of finitely many row-operation matrices.*

(In other words, every non-singular matrix can be 'factorized' into row-operation matrices.)

14. Recall what Lemma (β) says (when put in plain words): the product of any two invertible matrices is an invertible. We now upgrade Lemma (β) with the help of Theorem (B), to obtain Lemma (β').

Lemma (β').

Let A, B be $(n \times n)$ -square matrices.

Suppose A, B are non-singular and invertible.

Then the product AB is non-singular and invertible.

15. The converse of Lemma (β'), as formulated below, is also true.

Lemma (η).

Let A, B be $(n \times n)$ -square matrices.

Suppose the product AB is non-singular and invertible.

Then each of A, B is non-singular and invertible.

Proof of Lemma (η).

Suppose AB is non-singular and invertible.

We verify that B is non-singular:

- [This amounts to verifying the statement ‘for any $\mathbf{v} \in \mathbb{R}^n$, if $B\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$.’]

Pick any $\mathbf{v} \in \mathbb{R}^n$. Suppose $B\mathbf{v} = \mathbf{0}$.

Then $(AB)\mathbf{v} = A(B\mathbf{v}) = A\mathbf{0} = \mathbf{0}$.

Since AB is non-singular and $(AB)\mathbf{v} = \mathbf{0}$, we have $\mathbf{v} = \mathbf{0}$.

It follows that B is non-singular.

Now, by Theorem (B), the matrix B is invertible, with matrix inverse B^{-1} . Note that B^{-1} is invertible.

Note that $A = AI_n = A(BB^{-1}) = (AB)B^{-1}$.

Since AB is invertible and B^{-1} is invertible, A is also invertible according to Lemma (β).

Now, by Theorem (B), the matrix A is non-singular.

Remark.

We may combine Lemma (β') and Lemma (η) to obtain Theorem (D).

16. **Theorem (D).**

Suppose A, B are $(n \times n)$ -square matrices.

Then the statements below are logically equivalent:

(#) Each of A, B is non-singular and invertible.

(b) The product AB is non-singular and invertible.

17. **Corollary to Theorem (D).**

Suppose B_1, B_2, \dots, B_k are $(n \times n)$ -square matrices.

Then the statements below are logically equivalent:

(##) Each of B_1, B_2, \dots, B_k is non-singular and invertible.

(bb) The product $B_1 B_2 \cdots B_k$ is non-singular and invertible.

Proof of Corollary to Theorem (D).

Apply Theorem (D) and mathematical induction.