1. Reminder on notations: $\mathbb{R}^n$ stands for the set of all (column) vectors with n entries.

   Recall the definition for the notion of null space:

   Let A be an $(m \times n)$-matrix. The null space of A is defined to be the set $\{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \mathbf{0}\}$. It is denoted by $\mathcal{N}(A)$.

   In terms of system of linear equations, $\mathcal{N}(A)$ is the solution set of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

2. **Definition. (Non-singular matrices and singular matrices.)**

   Let C be a $(p \times p)$-square matrix.

   (a) *C is said to be non-singular if $\mathcal{N}(C) = \{\mathbf{0}\}$.*

   (b) *C is said to be singular if C is not non-singular.*

   Remarks.

   (a) We don't talk about non-singularity for non-square matrices.

   (b) The statement '$\mathcal{N}(C) = \{\mathbf{0}\}$' in the context of this definition for the notion of *non-singular matrices* is a set equality.

   The correct (and formal) way to understand the equality '$\mathcal{N}(C) = \{\mathbf{0}\}$' is that it is a 'short-hand' for this passage:

   > *Both statements (†), (‡) are true:*
   >
   > (†) *For any $\mathbf{x} \in \mathbb{R}^p$, if $\mathbf{x} \in \mathcal{N}(C)$ then $\mathbf{x} \in \{\mathbf{0}\}$.*
   >
   > (‡) *For any $\mathbf{x} \in \mathbb{R}^p$, if $\mathbf{x} \in \{\mathbf{0}\}$ then $\mathbf{x} \in \mathcal{N}(C)$.*

   Because '$\mathbf{x} \in \{\mathbf{0}\}$' is just a (clumsy) re-formulation of '$\mathbf{x} = \mathbf{0}$', the statement (‡) is trivially true by virtue of the properties of matrix multiplication, and it can be safely ignored. The essential mathematical content in the statement '$\mathcal{N}(C) = \{\mathbf{0}\}$' is the statement (†).

   (c) There are various (direct) re-formulations (according to definition) for the statement '*the $(p \times p)$-square matrix C is non-singular*'. Dependent on the concrete problem we are dealing with, one of them may be much easier to use than any other.

3. **Lemma (1). (Simple re-formulations of the notion of non-singularity.)**

   Let C be a $(p \times p)$-square matrix. The statements below are logically equivalent:

   ($\diamondsuit$) *$\mathcal{N}(C) = \{\mathbf{0}\}$.*

   ($\clubsuit$) *$\mathbf{0}$ is the only vector in the null space of C.*

   ($\heartsuit$) *For any vector $\mathbf{v} \in \mathbb{R}^p$, if $C\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$.*

   ($\spadesuit$) *The trivial solution is the only solution for the homogeneous system $\mathcal{LS}(C, \mathbf{0})$.*

   Remark. Corresponding to the statement Lemma (1), we may give various (direct) re-formulations (according to definition) for the statement '*the $(p \times p)$-square matrix C is singular*'. Dependent on the concrete problem we are dealing with, one of them may be much easier to use than any other:

   > *Let C be a $(p \times p)$-square matrix. The statements below are logically equivalent:*
   >
   > ($\sim\diamondsuit$) *$\mathcal{N}(C) \neq \{\mathbf{0}\}$.*
   >
   > ($\sim\clubsuit$) *There is a non-zero vector in the null space of C.*
   >
   > ($\sim\heartsuit$) *There is some $\mathbf{v} \in \mathbb{R}^p$ such that $\mathbf{v} \neq \mathbf{0}$ and $C\mathbf{v} = \mathbf{0}$.*
   >
   > ($\sim\spadesuit$) *There is some non-trivial solution for the homogeneous system $\mathcal{LS}(C, \mathbf{0})$.*

4. **Examples of non-singular matrices.**

   (a) *Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$. We verify that A is non-singular.*

   - What to check? '$\mathcal{N}(A) = \{\mathbf{0}\}$'.
   - What easiest to check? 'The trivial solution is the only solution for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$'.
   - Detail of argument:

     We find the reduced row-echelon form A' which is row-equivalent to A by applying row operations:

     $$A \longrightarrow \cdots \to A' = I_2.$$

     (Fill in the detail of the calculations.)
     It follows that the only solution for $\mathcal{LS}(A, \mathbf{0})$ is the trivial solution '$\mathbf{x} = \mathbf{0}$'.
     Hence A is non-singular.

(b) *Let* $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix}$. *We verify that A is non-singular.*

- What to check? '$\mathcal{N}(A) = \{\mathbf{0}\}$'.
- What easiest to check? 'The trivial solution is the only solution for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$'. How to proceed?

(c) *Let* $A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$. *We verify that A is non-singular.*

- What to check? What easiest to check? How to proceed?

5. **Examples of singular matrices.**

(a) *Let* $A = \begin{bmatrix} 1 & -5 & 3 \\ 2 & -4 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. *We verify that A is singular.*

- What to check? '$\mathcal{N}(A) \neq \{\mathbf{0}\}$'.
- What easiest to check? 'There is a non-trivial solution for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$'.
- Detail of argument:

We find the reduced row-echelon form A' which is row-equivalent to A by applying row operations:

$$A \longrightarrow \cdots \rightarrow A' = \begin{bmatrix} 1 & 0 & -7/6 \\ 0 & 1 & -5/6 \\ 0 & 0 & 0 \end{bmatrix}$$

(Fill in the detail of the calculations.)

It follows that '$\mathbf{x} = \begin{bmatrix} 7/6 \\ 5/6 \\ 1 \end{bmatrix}$' is a non-trivial solution for $\mathcal{LS}(A, \mathbf{0})$.

Hence A is singular.

(b) *Let* $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. *We verify that A is singular.*

- What to check? '$\mathcal{N}(A) \neq \{\mathbf{0}\}$'.
- What easiest to check? 'There is a non-trivial solution for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$'. How to proceed?

(c) *Let* $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 2 & 7 & -12 \end{bmatrix}$. *We verify that A is singular.*

- What to check? What easiest to check? How to proceed?

6. **Lemma (2). (Sufficiency criterion for non-singularity in terms of matrix multiplication.)**

Let C be a $(p \times p)$-square matrix.

Suppose there exists some $(p \times p)$-square matrix J such that $JC = I_p$.

Then C is non-singular.

Proof of Lemma (2).

Let C be a $(p \times p)$-square matrix.

Suppose there exists some $(p \times p)$-square matrix J such that $JC = I_p$.

[Ask: What to check? 'C is non-singular'.
Which formulation is easiest to use? 'For any $\mathbf{v} \in \mathbb{R}^p$, if $C\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$.'
Now ask: How to proceed?]

Pick any $\mathbf{v} \in \mathbb{R}^p$. Suppose $C\mathbf{v} = \mathbf{0}$. [Try to deduce: '$\mathbf{v} = \mathbf{0}$.']

By assumption $JC = I_p$. Then $(JC)\mathbf{v} = I_p\mathbf{v} = \mathbf{v}$.

Recall that $C\mathbf{v} = \mathbf{0}$. Then $\mathbf{v} = (JC)\mathbf{v} = J(C\mathbf{v}) = J\mathbf{0} = \mathbf{0}$.

[We have successfully deduced 'For any $\mathbf{v} \in \mathbb{R}^p$, if $C\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$.']

It follows that C is non-singular.

7. **Natural questions to ask, as follow-up to Lemma (2).**

(a) The converse of Lemma (2) reads:

Let C be a $(p \times p)$-square matrix.
Suppose C is non-singular.
Then there exists some $(p \times p)$-square matrix J such that $JC = I_p$.

Question. *Is the converse of Lemma (2) true?*

Answer. It will turn out to be a true statement. (But to see this, a lot of work needs to be done first.)

(b) The statement ($\sharp$) is a generalization of Lemma (2):

($\sharp$) *Let C be a $(p \times q)$-square matrix.*
Suppose there exists some $(q \times p)$-square matrix J such that $JC = I_q$.
Then $\mathcal{N}(C) = \{\mathbf{0}\}$.

Question. *Is the statement ($\sharp$) true?*

Answer. Yes. (How to prove the answer? Exercise.)

8. **More examples of non-singular matrices.**

(a) I_n is non-singular.

(b) Every permutation matrix is non-singular.

An $(n \times n)$-matrix for which there is exactly one 1 in each row and each column, and every other entry is 0 is called a permutation matrix.

Examples:

- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$

(c) Every orthogonal matrix is non-singular.

Why? Recall definition: *An $(n \times n)$-square matrix C is orthogonal if $C^tC = CC^t = I_n$.*

Now what does Lemma (1) say?

(d) Every upper uni-triangular matrix is non-singular. (Reason: Lemma (2).) Examples:

- $A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$.
- $B = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix}$.
- $C = \begin{bmatrix} 1 & \alpha & \beta & \gamma \\ 0 & 1 & \delta & \epsilon \\ 0 & 0 & 1 & \eta \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

A square matrix for which all diagonal entries are 1 and all entries below the diagonal are 0 is called an upper uni-triangular matrix.

9. **More examples of singular matrices.**

(a) The zero square matrix is singular.

(b) Every strictly upper triangular matrix is singular. Examples:

- $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$.
- $B = \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix}$.
- $C = \begin{bmatrix} 0 & \alpha & \beta & \gamma \\ 0 & 0 & \delta & \epsilon \\ 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

A square matrix for which all diagonal entries and all entries below the diagonal are 0 is called a strictly upper triangular matrix.

(c) Every square matrix with an entire column of 0's is singular.

Illustration through (4×4)-square matrices:

- Suppose $A = \begin{bmatrix} 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$. We claim that A is singular. How to see this?

Can we name a non-zero vector $\mathbf{v}$ in $\mathbb{R}^4$ for which $A\mathbf{v} = \mathbf{0}$?

Yes, we take $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Then $A\mathbf{v} = \mathbf{0}$.

- How about $A = \begin{bmatrix} * & 0 & * & * \\ * & 0 & * & * \\ * & 0 & * & * \\ * & 0 & * & * \end{bmatrix}$? Or $A = \begin{bmatrix} * & * & 0 & * \\ * & * & 0 & * \\ * & * & 0 & * \\ * & * & 0 & * \end{bmatrix}$? Or $A = \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \end{bmatrix}$?

Question. How about an $(n \times n)$-square matrices whose entries in the j-th column are all 0?

(d) Every square matrix with an entire row of 0's is singular.

Illustration through (4×4)-matrices:

- Suppose $A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$. We claim that A is singular. How to see this?

  Apply Gaussian elimination
$$A \longrightarrow ... \longrightarrow A'$$
  to obtain the reduced row-echelon form A' which is row-equivalent to A.

  The bottom row of A' is a row of 0's. So $A' = \begin{bmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

  The rank of A' is at most 3. The homogeneous system $\mathcal{LS}(A', \mathbf{0})$ will have a non-trivial solution, say, '$\mathbf{x} = \mathbf{v}$', which will also be a non-trivial solution of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.
  Therefore A is singular.

- How about $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$? Or $A = \begin{bmatrix} * & * & * & * \\ 0 & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$? Or $A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 0 \\ * & * & * & * \end{bmatrix}$?

Question. How about an $(n \times n)$-square matrices whose entries in the i-th row are all 0?

10. **Lemma (3). (Special role of identity matrix amongst reduced row-echelon forms.)**

Let A be an $(n \times n)$-square matrix.

Suppose A is a reduced row-echelon form.

Then A is non-singular if and only if $A = I_n$.

Remark. Lemma (3) tells us that I_n is the only $(n \times n)$-square matrix which is simultaneously a reduced row-echelon form and a non-singular matrix. Every reduced row-echelon form which is not I_n is singular.

Proof of Lemma (3).

Let A be an $(n \times n)$-square matrix.

Suppose A is a reduced row-echelon form.

- Suppose $A = I_n$. Then A is non-singular.
- Suppose A is non-singular.

  Note that there are r pivot columns in A, where r is the rank of A. By definition, $r \leq n$.

  Then A reads as
$$\left[\begin{array}{ccccccc} 1 & \cdots & 0 & \cdots\cdots & 0 & \cdots \\ 0 & \cdots & 1 & \cdots\cdots & 0 & \cdots \\ \vdots & & \vdots & & \vdots & \\ 0 & \cdots & 0 & \cdots\cdots & 1 & \cdots \\ \hline & \cdots & & \text{all 0's} & & \cdots \\ & \vdots & & \vdots & & \vdots \\ & \cdots & & \text{all 0's} & & \cdots \end{array}\right]$$

  We claim that $r = n$:

  Suppose it were true that $r < n$. Then, because the number of leading ones is strictly smaller than the number of columns, it would happen that some columns of A would fail to be a pivot column. Furthermore, because there is the same number of rows as of columns, some rows of A would fail to contain a leading one.
  Now it would happen that there was at least one row of 0's in A. Then A would be singular. Contradiction arises.

  Therefore $r = n$ is the only possibility. Then each column of A is a pivot column. Hence $A = I_n$.