

0. Reminder on notations:  $\mathbb{R}^n$  stands for the set of all (column) vectors with  $n$  entries.

1. **Definition. (Null space of a matrix.)**

Let  $A$  be an  $(m \times n)$ -matrix.

- (a) The system of linear equations  $\mathcal{LS}(A, \mathbf{0})$  is called the homogeneous system with coefficient matrix  $A$ .
- (b) The solution set of the homogeneous system  $\mathcal{LS}(A, \mathbf{0})$  is called the null space of  $A$ . It is denoted by  $\mathcal{N}(A)$ .

**Remark.** First of all, recall that

‘ $\mathbf{x} = \mathbf{u}$ ’ is a solution of  $\mathcal{LS}(A, \mathbf{0})$  if and only if  $A\mathbf{u} = \mathbf{0}$ .

Using as ‘selection criterion’ the equality ‘ $A\mathbf{x} = \mathbf{0}$ ’, we may present the null space of  $A$  as a set constructed using the method of specification:

- Those vectors in  $\mathbb{R}^n$  which, upon substitution into the ‘ $\mathbf{x}$ ’ in this ‘selection criterion’ result in an equality, will be ‘collected’.
- Those vectors in  $\mathbb{R}^n$  which, upon substitution into the ‘ $\mathbf{x}$ ’ in this ‘selection’ do not result in an equality, will be ‘discarded’.

Hence the null space of  $A$  is the set  $\{\mathbf{u} \in \mathbb{R}^n : A\mathbf{u} = \mathbf{0}\}$ .

**Further remark.** How to use the various versions of the definitions?

Always remember, whenever  $\mathbf{v} \in \mathbb{R}^n$ , the statements below mean the same thing:

- (a)  $\mathbf{v} \in \mathcal{N}(A)$ .
- (b)  $A\mathbf{v} = \mathbf{0}$ .
- (c) ‘ $\mathbf{x} = \mathbf{v}$ ’ is a solution of the homogeneous system  $\mathcal{LS}(A, \mathbf{0})$  with unknown  $\mathbf{x}$ .

To determine  $\mathcal{N}(A)$  is the same as giving an ‘explicit’ description of the solution set of the homogeneous system  $\mathcal{LS}(A, \mathbf{0})$  through set language, in terms of (hopefully just a few) solutions of the system. That amounts to finding all solutions of  $\mathcal{LS}(A, \mathbf{0})$ .

2. **Example (★).**

Determine the null space of the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}$$

explicitly (in terms of concrete vectors in  $\mathbb{R}^7$ ).

- (a) First determine the reduced row-echelon form  $A'$  which is row-equivalent to  $A$  by applying a sequence of row operations, say, Gaussian elimination, to the augmented matrix representation of  $\mathcal{LS}(A, \mathbf{0})$ :

$$[A|\mathbf{0}] \longrightarrow \dots \longrightarrow [A'|\mathbf{0}]$$

We find that

$$A' = \begin{bmatrix} 1 & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 1 & 2 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- (b)  $\mathcal{LS}(A', \mathbf{0})$  reads:

$$\begin{cases} x_1 + 4x_2 & & + 2x_5 + x_6 - 3x_7 = 0 \\ & x_3 & + x_5 - 3x_6 + 5x_7 = 0 \\ & & x_4 + 2x_5 - 6x_6 + 6x_7 = 0 \\ & & & & & & 0 = 0 \end{cases}$$

The solutions of  $\mathcal{LS}(A', \mathbf{0})$ , and hence of  $\mathcal{LS}(A, \mathbf{0})$ , are given by  $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$ , where

$$c_1, c_2, c_3, c_4 \text{ are arbitrary numbers, in which } \mathbf{u}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

This amounts to saying that for any  $\mathbf{v} \in \mathbb{R}^7$ , ‘ $\mathbf{x} = \mathbf{v}$ ’ is a solution of  $\mathcal{LS}(A, \mathbf{0})$  if and only if there exist some  $c_1, c_2, c_3, c_4 \in \mathbb{R}$  such that  $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$ .

(c) We now apply the method of specification to present the solution set of  $\mathcal{LS}(A, \mathbf{0})$  explicitly in terms of  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ , through the ‘selection criterion’

(†) ‘there exist some  $c_1, c_2, c_3, c_4 \in \mathbb{R}$  such that  $\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$ ’

in which  $\mathbf{y}$  is the ‘variable’.

How does the method work? Remember:

- Those and only those vectors in  $\mathbb{R}^7$  which upon substitution into the symbol  $\mathbf{y}$  in (†) turn it into a true statement will be collected.
- The others will be ‘discarded’.

So  $\mathcal{N}(A)$  is the set

$$\left\{ \mathbf{y} \in \mathbb{R}^7 : \begin{array}{l} \text{there exist some } c_1, c_2, c_3, c_4 \in \mathbb{R} \\ \text{such that } \mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 \end{array} \right\}$$

(As shorthand we present  $\mathcal{N}(A)$  as  $\{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 \mid c_1, c_2, c_3, c_4 \in \mathbb{R}\}$ .)

(d) **Comment on the presentation of the manipulations.**

During the manipulation

$$[A|\mathbf{0}] \longrightarrow \dots \longrightarrow [A'|\mathbf{0}]$$

we observe that the last column in every matrix in this sequence stays ‘ $\mathbf{0}$ ’. This is expected: no matter which row-operation is applied on the zero vector, it only convert the zero vector to itself.

Hence we can actually save time (and ink) by omitting the  $\mathbf{0}$ ’s throughout, and simply write

$$A \longrightarrow \dots \longrightarrow A'$$

provided we remember we are apply row operations on the coefficient matrices of various homogeneous system.

### 3. Examples on determining null space explicitly.

(a) Determine the null space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix}$$

Determine the reduced row-echelon form  $A'$  which is row-equivalent to  $A$  by applying a sequence of row operations to  $A$ :

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix} \xrightarrow{-1R_1+R_2} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 2 & 6 & 5 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \\ &\xrightarrow{-1R_3} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-1R_3+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A' \end{aligned}$$

The null space  $\mathcal{N}(A)$  of the matrix  $A$  is the solution set of  $\mathcal{LS}(A, \mathbf{0})$ , and hence is the solution set of  $\mathcal{LS}(A', \mathbf{0})$  as well. Note that  $\mathcal{LS}(A', \mathbf{0})$  reads:

$$\begin{cases} x_1 & & = 0 \\ & x_2 & = 0 \\ & & x_3 = 0 \end{cases}$$

The only solution of  $\mathcal{LS}(A, \mathbf{0})$  is given by  $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ .

Hence  $\mathcal{N}(A)$  is the set  $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ .

(b) Determine the null space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix}$$

Determine the reduced row-echelon form  $A'$  which is row-equivalent to  $A$  by applying a sequence of row operations to  $A$ :

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix} \xrightarrow{-1R_1+R_2} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 2 & 6 & 5 & 6 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -4 \end{bmatrix} \\ &\xrightarrow{-1R_3} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-1R_3+R_2} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix} = A' \end{aligned}$$

The null space  $\mathcal{N}(A)$  of the matrix  $A$  is the solution set of  $\mathcal{LS}(A, \mathbf{0})$ , and hence is the solution set of  $\mathcal{LS}(A', \mathbf{0})$  as well. Note that  $\mathcal{LS}(A', \mathbf{0})$  reads:

$$\begin{cases} x_1 & & & + & 2x_4 & = & 0 \\ & x_2 & & - & 3x_4 & = & 0 \\ & & x_3 & + & 4x_4 & = & 0 \end{cases}$$

The solutions of  $\mathcal{LS}(A, \mathbf{0})$  are given by  $\mathbf{x} = t\mathbf{u}$ , where  $t$  is an arbitrary number, in which  $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ -4 \\ 1 \end{bmatrix}$ .

Hence  $\mathcal{N}(A)$  is the set  $\{t\mathbf{u} \mid t \in \mathbb{R}\}$ .

(c) Determine the null space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix}$$

Determine the reduced row-echelon form  $A'$  which is row-equivalent to  $A$  by applying a sequence of row operations to  $A$ :

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix} \xrightarrow{-1R_1+R_2} \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 0 & -5 & 5 & -10 & -20 \end{bmatrix} \\ &\xrightarrow{-1R_2} \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & -5 & 5 & -10 & -20 \end{bmatrix} \xrightarrow{5R_2+R_3} \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A' \end{aligned}$$

The null space  $\mathcal{N}(A)$  of the matrix  $A$  is the solution set of  $\mathcal{LS}(A, \mathbf{0})$ , and hence is the solution set of  $\mathcal{LS}(A', \mathbf{0})$  as well. Note that  $\mathcal{LS}(A', \mathbf{0})$  reads:

$$\begin{cases} x_1 & & + & 2x_3 & - & 3x_4 & - & x_5 & = & 0 \\ & x_2 & - & x_3 & + & 2x_4 & + & 4x_5 & = & 0 \\ & & & & & & & 0 & = & 0 \end{cases}$$

The solutions of  $\mathcal{LS}(A, \mathbf{0})$  are given by  $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ , where  $c_1, c_2, c_3$  are arbitrary numbers, in

$$\text{which } \mathbf{u}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence  $\mathcal{N}(A)$  is the set  $\{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 \mid c_1, c_2, c_3 \in \mathbb{R}\}$ .

#### 4. Further consideration on Example (\*).

Let

$$A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}$$

Recall that  $\mathcal{N}(A)$  is the set  $\{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 \mid c_1, c_2, c_3, c_4 \in \mathbb{R}\}$ , in which  $\mathbf{u}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ,  $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ,

$$\mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) **Further question.**

What is so special about  $\mathcal{N}(A)$ , regarding its 'algebraic structure'?

**Answer to further question.**

The statements below hold:

- (1)  $\mathbf{0} \in \mathcal{N}(A)$ .
- (2) For any  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$ , if  $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$  then  $\mathbf{v} + \mathbf{w} \in \mathcal{N}(A)$ .

(3) For any  $\mathbf{v} \in \mathbb{R}^7$ , for any  $\alpha \in \mathbb{R}$ , if  $\mathbf{v} \in \mathcal{N}(A)$  then  $\alpha\mathbf{v} \in \mathcal{N}(A)$ .

(4) For any  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$ , for any  $\alpha, \beta \in \mathbb{R}$ , if  $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$  then  $\alpha\mathbf{v} + \beta\mathbf{w} \in \mathcal{N}(A)$ .

(b) **Justification of (1), (2), (3) in answer to further question.**

To apply what we see about  $\mathcal{N}(A)$  in concrete terms?

(1) Note that  $\mathbf{0} = 0 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + 0 \cdot \mathbf{u}_3 + 0 \cdot \mathbf{u}_4$ , and  $0 \in \mathbb{R}$ . Then  $\mathbf{0} \in \mathcal{N}(A)$ .

(2) Pick any  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$ .

Suppose  $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$ . Then there exist some  $c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4 \in \mathbb{R}$  such that  $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$  and  $\mathbf{w} = d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + d_3\mathbf{u}_3 + d_4\mathbf{u}_4$ .

Then  $\mathbf{v} + \mathbf{w} = \dots = (c_1 + d_1)\mathbf{u}_1 + (c_2 + d_2)\mathbf{u}_2 + (c_3 + d_3)\mathbf{u}_3 + (c_4 + d_4)\mathbf{u}_4$ , and  $c_1 + d_1, c_2 + d_2, c_3 + d_3, c_4 + d_4 \in \mathbb{R}$ .

Therefore  $\mathbf{v} + \mathbf{w} \in \mathcal{N}(A)$ .

(3) Pick any  $\mathbf{v} \in \mathbb{R}^7$ . Pick any  $\alpha \in \mathbb{R}$ .

Suppose  $\mathbf{v} \in \mathcal{N}(A)$ . Then there exists some  $c_1, c_2, c_3, c_4 \in \mathbb{R}$  such that  $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$ .

Then  $\alpha\mathbf{v} = \dots = (\alpha c_1)\mathbf{u}_1 + (\alpha c_2)\mathbf{u}_2 + (\alpha c_3)\mathbf{u}_3 + (\alpha c_4)\mathbf{u}_4$ , and  $\alpha c_1, \alpha c_2, \alpha c_3, \alpha c_4 \in \mathbb{R}$ .

Therefore  $\alpha\mathbf{v} \in \mathcal{N}(A)$ .

(c) **Another justification of (1), (2), (3) in answer to further question.**

To apply the definition of null space? (This is a better method.)

(1) Note that  $A\mathbf{0} = \mathbf{0}$ . Then  $\mathbf{0} \in \mathcal{N}(A)$ .

(2) Pick any  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$ .

Suppose  $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$ . Then  $A\mathbf{v} = \mathbf{0}$  and  $A\mathbf{w} = \mathbf{0}$ .

Therefore  $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}$ .

Hence  $\mathbf{v} + \mathbf{w} \in \mathcal{N}(A)$ .

(3) Pick any  $\mathbf{v} \in \mathbb{R}^7$ . Pick any  $\alpha \in \mathbb{R}$ .

Suppose  $\mathbf{v} \in \mathcal{N}(A)$ . Then  $A\mathbf{v} = \mathbf{0}$ .

Therefore  $A(\alpha\mathbf{v}) = \alpha A\mathbf{v} = \alpha \cdot \mathbf{0} = \mathbf{0}$ .

Hence  $\alpha\mathbf{v} \in \mathcal{N}(A)$ .

**Remark.** This ‘second justification’ of (1), (2), (3) is superior to the ‘first’, in the sense that almost nothing about explicit features of  $A$ , apart from the fact that it has 7 columns, is involved in the mathematical argument. We may wonder the mathematical reasoning in this ‘second justification’ may work when  $A$  is replaced by a general matrix. It turns out to be the case.

5. **Theorem (1). (Null space of a matrix as a ‘subspace’.)**

Let  $A$  be an  $(m \times n)$ -matrix. The statement below hold:

(1)  $\mathbf{0} \in \mathcal{N}(A)$ .

(2) For any  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ , if  $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$  then  $\mathbf{v} + \mathbf{w} \in \mathcal{N}(A)$ .

(3) For any  $\mathbf{v} \in \mathbb{R}^n$ , for any  $\alpha \in \mathbb{R}$ , if  $\mathbf{v} \in \mathcal{N}(A)$  then  $\alpha\mathbf{v} \in \mathcal{N}(A)$ .

(4) For any  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ , for any  $\alpha, \beta \in \mathbb{R}$ , if  $\mathbf{v}, \mathbf{w} \in \mathcal{N}(A)$  then  $\alpha\mathbf{v} + \beta\mathbf{w} \in \mathcal{N}(A)$ .

**Proof.** Exercise. (Extract what we did in the study of Example (★).)

**Remark.** We can further deduce that

For any  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k \in \mathbb{R}^n$ , for any  $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$ , if  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k \in \mathcal{N}(A)$  then  $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \dots + \alpha_k\mathbf{u}_k \in \mathcal{N}(A)$ .

In plain words (and in terms of the notion of *linear combinations*, to be introduced later), this amounts to saying:

Every linear combination of vectors in  $\mathcal{N}(A)$  is a vector in  $\mathcal{N}(A)$ .

6. **Reformulation of Theorem (1) in terms of homogeneous systems.**

Let  $A$  be an  $(m \times n)$ -matrix. The statement below hold:

(1) ‘ $\mathbf{x} = \mathbf{0}$ ’ is a solution of  $\mathcal{LS}(A, \mathbf{0})$ .

(2) Let  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ . Suppose ‘ $\mathbf{x} = \mathbf{v}$ ’, ‘ $\mathbf{x} = \mathbf{w}$ ’ are solutions of  $\mathcal{LS}(A, \mathbf{0})$ . Then ‘ $\mathbf{x} = \mathbf{v} + \mathbf{w}$ ’ is a solution of  $\mathcal{LS}(A, \mathbf{0})$ .

(3) Let  $\mathbf{v} \in \mathbb{R}^n$ . Let  $\alpha \in \mathbb{R}$ . Suppose ‘ $\mathbf{x} = \mathbf{v}$ ’ is a solution of  $\mathcal{LS}(A, \mathbf{0})$ . Then ‘ $\mathbf{x} = \alpha\mathbf{v}$ ’ is a solution of  $\mathcal{LS}(A, \mathbf{0})$ .

(4) Let  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ . Let  $\alpha, \beta \in \mathbb{R}$ . Suppose ‘ $\mathbf{x} = \mathbf{v}$ ’, ‘ $\mathbf{x} = \mathbf{w}$ ’ are solutions of  $\mathcal{LS}(A, \mathbf{0})$ . Then ‘ $\mathbf{x} = \alpha\mathbf{v} + \beta\mathbf{w}$ ’ is a solution of  $\mathcal{LS}(A, \mathbf{0})$ .