MATH1030 The use of set notations in linear algebra.

0. In linear algebra very often we deal not with individual vectors (or individual matrices), but collections of various vectors (or collections of various matrices).

   In handling such collections, we use the language of sets.

   We are going to introduce set notations in the context of sets of vectors (of $\mathbb{R}^n$ for the same n). However, the discussion here be adapted to sets of other types of objects (say, matrices, linear transformations).

1. **The notions of 'belong to', 'element of a set (of vectors)'.**

   Suppose we have 'collected' (possibly many) vectors, and we think of this collection as an entity on its own. Then we will refer to such an entity as a *set*.

   (a) Label this set by, say, S. Suppose some vector $\mathbf{u}$ is amongst the vectors 'collected inside' the set S.

   (b) Then we will agree to say that *the vector* $\mathbf{u}$ *belongs to the set* S (or $\mathbf{u}$ *is an element of the set* S.) As a short-hand, we may write

   $$\text{`}\mathbf{u} \in S\text{'}$$

   (c) If it happens that some vector $\mathbf{t}$ in $\mathbb{R}^n$ is not amongst the vectors collected inside the set S then we may write

   $$\text{`}\mathbf{t} \notin S\text{'}$$

   (which reads '$\mathbf{t}$ *does not belong to* S', or '$\mathbf{t}$ *is not an element of the set* S').

   For each positive integer n, the collection of all (column) vectors with n (real) entries constitute a set which is labelled $\mathbb{R}^n$.
   When we write '$\mathbf{v} \in \mathbb{R}^n$', we mean

   '$\mathbf{v}$ *is a (column) vector with n (real) entries.*'

2. **Notations and terminologies for 'small' sets (of vectors).**

   Suppose $\mathbf{u}, \mathbf{v}, \cdots, \mathbf{w}$ are 'finitely many' vectors in $\mathbb{R}^n$, (in the sense that we can list them out exhaustively).

   Then we agree that we may present the set whose elements are exactly $\mathbf{u}, \mathbf{v}, \cdots, \mathbf{w}$ as

   $$\{\mathbf{u}, \mathbf{v}, \cdots, \mathbf{w}\}.$$

   We call this entity *the set whose elements are the vectors* $\mathbf{u}, \mathbf{v}, \cdots, \mathbf{w}$.

   The symbol '{' signifies the beginning of the list of vectors.

   The symbol '}' signifies the end of the list of vectors.

3. **Conventions on the notations for 'small' sets.**

   (a) 'Repetition in the list' does not count.
   Provided a vector, say, $\mathbf{u}$, is an element of a 'small' set, say, S, $\mathbf{u}$ has to be presented at least once in the list representing S. However, no matter how many more times $\mathbf{u}$ is presented, it still counts as once.
   Example: $\{\mathbf{u}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{v}, \mathbf{v}\}$ is the same set as $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

   (b) 'Ordering in the list' does not matter.
   Given any two lists, as long every vector which is presented in each list is also presented in the other, the two lists will represent the same 'small' set, regardless of the order in which the vectors is presented in each list.
   Example:
   $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, $\{\mathbf{v}, \mathbf{w}, \mathbf{u}\}$, $\{\mathbf{w}, \mathbf{u}, \mathbf{v}\}$, $\{\mathbf{u}, \mathbf{w}, \mathbf{v}\}$, $\{\mathbf{v}, \mathbf{u}, \mathbf{w}\}$, $\{\mathbf{w}, \mathbf{v}, \mathbf{u}\}$ all stand for the same set of vectors, namely the set whose elements are exactly the not necessarily distinct vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

   (c) The set to which no vector, nor anything else, belongs is denoted by $\emptyset$, and is called the empty set.

4. **Example.**

   The chain of symbols '$\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix}, \begin{bmatrix}3\\0\\5\end{bmatrix} \right\}$' stands for the set with exactly four elements, namely the vectors $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix}, \begin{bmatrix}3\\0\\5\end{bmatrix}$.

   When we denote this set by S, we have:

   $$\begin{bmatrix}1\\2\\3\end{bmatrix} \in S, \quad \begin{bmatrix}2\\3\\1\end{bmatrix} \in S, \quad \begin{bmatrix}0\\2\\4\end{bmatrix} \in S, \quad \begin{bmatrix}3\\0\\5\end{bmatrix} \in S.$$

It happens that $\begin{bmatrix}0\\0\\0\end{bmatrix} \notin S$, and $\begin{bmatrix}3\\2\\1\end{bmatrix} \notin S$.

Note that the sets below are all S itself in disguise:

$$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix}, \begin{bmatrix}3\\0\\5\end{bmatrix}\right\}, \left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix}, \begin{bmatrix}3\\0\\5\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right\}, \left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\0\\5\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}\right\}, \left\{\begin{bmatrix}2\\3\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\0\\5\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix}, \begin{bmatrix}3\\0\\5\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}\right\}.$$

5. **Method of specification for sets (of vectors).**

Many a set of vectors cannot be presented as a list of finitely many vectors, because there are too many vectors to exhaustively write down.

Sometimes, even though a set of vectors in $\mathbb{R}^n$ may be presented as a list, for one reason or other we may choose not to do so.

In this situation we can pinpoint the set concerned by writing down an appropriate 'selection criterion' (called *predicates*) which instructs ourselves exactly which vectors are to be collected (and which not).

This method of describing a set is called the method of specification.

6. **Example of Method of specification: solution sets of systems of linear equations.**

Suppose A is an $(m \times n)$-matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^m$.

Suppose we want to collect into a set those and only those vectors in $\mathbb{R}^n$, which upon multiplication by the matrix A from the left, will result in the vector $\mathbf{b}$.

Then an appropriate 'selection criterion' will be

'$A\mathbf{x} = \mathbf{b}$',

in which $\mathbf{x}$ is called the 'variable' in this 'selection criterion'.

As A is a $(m \times n)$-matrix, only vectors from $\mathbb{R}^m$ will be considered for selection.

- Those vectors in $\mathbb{R}^n$ which, upon substitution into the '$\mathbf{x}$' in this 'selection criterion' result in an equality, will be 'collected'.
- Those vectors in $\mathbb{R}^n$ which, upon substitution into the '$\mathbf{x}$' in this 'selection' do not result in an equality, will be 'discarded'.

The resultant collection, which is the set we are looking for, will be expressed as

$$\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$$

This set is the collection of all solutions of $\mathcal{LS}(A, \mathbf{b})$. Hence it is called the solution set of the system $\mathcal{LS}(A, \mathbf{b})$.

Comments.

(a) The symbol '$\mathbf{x}$' in this chain of symbols is a dummy. The set can be expressed as $\{\mathbf{y} \in \mathbb{R}^n : A\mathbf{y} = \mathbf{b}\}$, or $\{\mathbf{z} \in \mathbb{R}^n : A\mathbf{z} = \mathbf{b}\}$.

(b) Always remember: given any $\mathbf{u} \in \mathbb{R}^n$, the vector $\mathbf{u}$ belongs to this set if and only if the equality $A\mathbf{u} = \mathbf{b}$ holds.

7. **Example of Method of specification: 'column spaces' of matrices.**

Suppose H is an $(p \times q)$-matrix.

Suppose we want to collect into a set those and only those vectors in $\mathbb{R}^p$ that can be expressed in the form $H\mathbf{v}$ in which $\mathbf{v}$ is a vector in $\mathbb{R}^q$.

Then an appropriate 'selection criterion' will be

'*there exist some* $\mathbf{v} \in \mathbb{R}^q$ *such that* $\mathbf{x} = H\mathbf{v}$',

in which $\mathbf{x}$ is called the 'variable' in this 'selection criterion'.

As H is a $(p \times q)$-matrix, only vectors from $\mathbb{R}^p$ will be considered for selection.

- Those vectors in $\mathbb{R}^p$ which, upon substitution into the '$\mathbf{x}$' in this 'selection criterion' result in a true statement, will be 'collected'.
- Those vectors in $\mathbb{R}^p$ which, upon substitution into the '$\mathbf{x}$' in this 'selection' do not result in a true statement, will be 'discarded'.

The resultant collection, which is the set we are looking for, will be expressed as

$$\{\mathbf{x} \in \mathbb{R}^p : \textit{ There exists some } \mathbf{v} \in \mathbb{R}^q \textit{ such that } \mathbf{x} = H\mathbf{v} \ \}.$$

This set will be called the column space of the matrix H.

Comments.

(a) The symbols '$\mathbf{x}$', '$\mathbf{v}$' in this chain of symbols are dummies. The set can be expressed as

$$\{\mathbf{y} \in \mathbb{R}^p : \text{ There exists some } \mathbf{v} \in \mathbb{R}^q \text{ such that } \mathbf{y} = H\mathbf{v} \ \},$$

or

$$\{\mathbf{y} \in \mathbb{R}^p : \text{ There exists some } \mathbf{w} \in \mathbb{R}^q \text{ such that } \mathbf{y} = H\mathbf{w} \ \}.$$

(b) Some people to choose to present the 'selection criterion'

'*there exist some* $\mathbf{v} \in \mathbb{R}^q$ *such that* $\mathbf{x} = H\mathbf{v}$',

as

'$\mathbf{x} = H\mathbf{v}$ *for some* $\mathbf{v} \in \mathbb{R}^q$'.

So this set also presented as

$$\{\mathbf{y} \in \mathbb{R}^p : \ \mathbf{y} = H\mathbf{v} \text{ for some } \mathbf{v} \in \mathbb{R}^q \ \}.$$

(c) Some people to choose to highlight the the chain of symbols '$\mathbf{v} \in R^q$' and '$H\mathbf{v}$' in the 'selection criterion'

'*there exist some* $\mathbf{v} \in \mathbb{R}^q$ *such that* $\mathbf{x} = H\mathbf{v}$',

and abbreviate this set as

$$\{H\mathbf{v} \mid \mathbf{v} \in \mathbb{R}^q\}.$$

(d) Always remember: given any $\mathbf{z} \in \mathbb{R}^p$, the vector $\mathbf{z}$ belongs to this set if and only if there exists some $\mathbf{v} \in \mathbb{R}^q$ such that $\mathbf{z} = H\mathbf{v}$.

8. We will encounter more sets (of vectors) constructed with the method of specification. They will be introduced when we indeed need them. The way they are to be comprehended will be the same as the examples above.

9. **Solving systems of linear equations, revisited.**

Suppose A is an $(m \times n)$-matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^m$.

To solve the system of linear equations, say, $\mathcal{LS}(A, \mathbf{b})$, is the same as to provide an 'explicit' description (in set language) of its solution set $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$.

We are going to illustrate what this means through examples, recycled from the handout *What is solving a system of linear equations*. The discussion in the examples will apply in the general situation.

(a) Refer to Example (1) in the handout *What is solving a system of linear equations*.

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

After some work, we find that the only solution of the system $\mathcal{LS}(A, \mathbf{b})$ is $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

We may present this conclusion as:

The solution set of $\mathcal{LS}(A, \mathbf{b})$ *is given by* $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$.

(b) Refer to Example (2) in the handout *What is solving a system of linear equations*.

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

After some work, we find that the only solution of the system $\mathcal{LS}(A, \mathbf{b})$ is $\mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$.

We may present this conclusion as:

The solution set of $\mathcal{LS}(A, \mathbf{b})$ *is given by* $\left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \right\}$.

(c) Refer to Example (3) in the handout *What is solving a system of linear equations*.

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & -5 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}$.

After some work, we find that the system $\mathcal{LS}(A, \mathbf{b})$ has no solution.

We may present this conclusion as:

The solution set of $\mathcal{LS}(A, \mathbf{b})$ *is the empty set.*

(d) Refer to Example (4) in the handout *What is solving a system of linear equations*.

Let $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 2 & 7 & -12 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ -4 \\ 11 \end{bmatrix}$.

After some work, we find that the solutions of the system $\mathcal{LS}(A, \mathbf{b})$ are given by $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ where t is an arbitrary real number.

This amounts to saying that

$\mathbf{x}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$ if and only if there exists some $t \in \mathbb{R}$ such that $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

With the help of the method of specification, we may present this conclusion as:

The solution set of $\mathcal{LS}(A, \mathbf{b})$ is given by

$$\left\{ \mathbf{x} \in \mathbb{R}^3 : \textit{There exists some } t \in \mathbb{R} \textit{ such that } \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Or in short-hand:

The solution set of $\mathcal{LS}(A, \mathbf{b})$ is given by

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

(e) Refer to Example (5) in the handout *What is solving a system of linear equations.*

Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$.

After some work, we find that the solutions of the system $\mathcal{LS}(A, \mathbf{b})$ are given by $\mathbf{x} = \begin{bmatrix} -1 \\ 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

where s, t are arbitrary real numbers.

This amounts to saying that

$\mathbf{x}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$ if and only if there exists some $s, t \in \mathbb{R}$ such that $\mathbf{x} = \begin{bmatrix} -1 \\ 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$.

With the help of the method of specification, we may present this conclusion as:

The solution set of $\mathcal{LS}(A, \mathbf{b})$ is given by

$$\left\{ \mathbf{x} \in \mathbb{R}^4 : \textit{There exist some } s, t \in \mathbb{R} \textit{ such that } \mathbf{x} = \begin{bmatrix} -1 \\ 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Or in short-hand:

The solution set of $\mathcal{LS}(A, \mathbf{b})$ is given by

$$\left\{ \begin{bmatrix} -1 \\ 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}.$$

(f) Refer to Example (6) in the handout *What is solving a system of linear equations.*

Let $A = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 2 & 3 \\ -2 & -1 & -3 & 3 & 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$.

After some work, we find that the solutions of the system $\mathcal{LS}(A, \mathbf{b})$ are given by $\mathbf{x} = \begin{bmatrix} 10 \\ -8 \\ 0 \\ 5 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

where s, t are arbitrary real numbers.

This amounts to saying that

$\mathbf{x}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$ if and only if there exists some $s, t \in \mathbb{R}$ such that $\mathbf{x} = \begin{bmatrix} 10 \\ -8 \\ 0 \\ 5 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

With the help of the method of specification, we may present this conclusion as:

The solution set of $\mathcal{LS}(A, \mathbf{b})$ is given by

$$\left\{ \mathbf{x} \in \mathbb{R}^5 : \textit{There exist some } s, t \in \mathbb{R} \textit{ such that } \mathbf{x} = \begin{bmatrix} 10 \\ -8 \\ 0 \\ 5 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Or in short-hand:

The solution set of $\mathcal{LS}(A,\, \mathbf{b})$ is given by

$$\left\{ \begin{bmatrix} 10 \\ -8 \\ 0 \\ 5 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}.$$

10. **Definition for the notion of set equality (for sets of vectors).**

In general, we say that two sets (of vectors), say, K, L, are equal to each other, and write $K = L$, exactly when every element of each of K, L belongs to the other set as well.

In symbols:

$K = L$ if and only if both of (†), (‡) are true:

(†) For any $\mathbf{u}$, if $\mathbf{u} \in K$ then $\mathbf{u} \in L$.

(‡) For any $\mathbf{v}$, if $\mathbf{v} \in L$ then $\mathbf{v} \in K$.

(In plain words, (†) reads '*every vector which belongs to K will belong to L as well*', whereas (‡) reads '*every vector which belongs to L will belong to K as well*'.)

11. **Illustrations of the use of set equality.**

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \right\}.$

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \right\}.$

(b) Refer to Example (2) in the handout *What is solving a system of linear equations.*

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

After some work, we find that the only solution of the system $\mathcal{LS}(A,\, \mathbf{b})$ is $\mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$.

We may present this conclusion as:

The solution set of $\mathcal{LS}(A,\, \mathbf{b})$ is given by $\left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \right\}$.

This is in fact a verbal description of the set equality

$$\big\{ \mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{b} \big\} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \right\}$$

(c) Refer to Example (3) in the handout *What is solving a system of linear equations.*

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & -5 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}$.

After some work, we find that the system $\mathcal{LS}(A,\, \mathbf{b})$ has no solution.

We may present this conclusion as:

The solution set of $\mathcal{LS}(A,\, \mathbf{b})$ is the empty set.

This is in fact a verbal description of the set equality

$$\big\{ \mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{b} \big\} = \emptyset.$$

(d) Refer to Example (4) in the handout *What is solving a system of linear equations.*

Let $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 2 & 7 & -12 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ -4 \\ 11 \end{bmatrix}$.

After some work, we find that the solutions of the system $\mathcal{LS}(A,\, \mathbf{b})$ are given by $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ where t is an arbitrary real number.

This amounts to saying that

$\mathbf{x}$ is a solution of $\mathcal{LS}(A,\,\mathbf{b})$ if and only if there exists some $t \in \mathbb{R}$ such that $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

With the help of the method of specification, we may present this conclusion as:

The solution set of $\mathcal{LS}(A,\,\mathbf{b})$ is given by

$$\left\{ \mathbf{x} \in \mathbb{R}^3 : \textit{There exists some } t \in \mathbb{R} \textit{ such that } \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Or in short-hand:

The solution set of $\mathcal{LS}(A,\,\mathbf{b})$ is given by

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

This is in fact a verbal description of the set equality

$$\left\{ \mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{b} \right\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

(e) Recall the definition for the notion of *equivalent systems* (now re-formulated with the help of matrices and vectors):

Let A, A' be $(m \times n)$-matrices, and $\mathbf{b}, \mathbf{b}'$ be vectors in $\mathbb{R}^m$.
We say $\mathcal{LS}(A,\,\mathbf{b})$ is equivalent to $\mathcal{LS}(A',\,\mathbf{b}')$ as systems if and only if both statements below hold:

i. *Every solution of $\mathcal{LS}(A,\,\mathbf{b})$ is a solution of $\mathcal{LS}(A,\,\mathbf{b})$.*
ii. *Every solution of $\mathcal{LS}(A',\,\mathbf{b}')$ is a solution of $\mathcal{LS}(A,\,\mathbf{b})$.*

Now, with the notions of solution set and set equality, this definition may be re-formulated as:

Let A, A' be $(m \times n)$-matrices, and $\mathbf{b}, \mathbf{b}'$ be vectors in $\mathbb{R}^m$.
We say $\mathcal{LS}(A,\,\mathbf{b})$ is equivalent to $\mathcal{LS}(A',\,\mathbf{b}')$ as systems if and only if

$$\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\} = \{\mathbf{y} \in \mathbb{R}^n : A'\mathbf{y} = \mathbf{b}'\}.$$