1. **Definition. (Matrix Multiplication.)**

(a) *Let A be a row vector with n entries, given by $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$, and B be a column vector with n entries, given by* $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

We define the product AB to be the (1×1)-matrix $[a_1b_1 + a_2b_2 + \cdots + a_nb_n]$.
For future convenience we abuse notations to confuse as the number $a_1b_1 + a_2b_2 + \cdots + a_nb_n$.

(b) *Let A be an $(m \times n)$-matrix, whose k-th row is denoted by A_k, and B be a column vector with n entries, given by* $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

We define the product AB to be the column vector with m entries, given by $AB = \begin{bmatrix} A_1B \\ A_2B \\ \vdots \\ A_mB \end{bmatrix}$. *(The k-th entry of AB is A_kB.)*

Remark. Denoting the (i, j)-th entry of A by a_{ij}, the k-th entry of AB is given by the number $\sum_{j=1}^{n} a_{kj}b_j = a_{k1}b_1 + a_{k2}b_2 + \cdots + a_{kn}b_n$.

We have $AB = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + \cdots + a_{2n}b_n \\ \vdots \\ a_{m1}b_1 + a_{m2}b_2 + \cdots + a_{mn}b_n \end{bmatrix}$.

(c) *Let A be an $(m \times n)$-matrix, and B be an $(n \times p)$-matrix, whose ℓ-th column is denoted by B_ℓ.*
We define the product AB to be the $(m \times p)$-matrix, given by $AB = \begin{bmatrix} AB_1 \mid AB_2 \mid \cdots \mid AB_p \end{bmatrix}$. (The ℓ-th column of AB is AB_ℓ.)

Remark. Denoting the (i, j)-th entry of A by a_{ij}, and the (k, ℓ)-th entry of B by $b_{k\ell}$, the (i, ℓ)-th entry of AB is given by the number $\sum_{j=1}^{n} a_{ij}b_{j\ell} = a_{i1}b_{1\ell} + a_{i2}b_{2\ell} + \cdots + a_{in}b_{n\ell}$.

Denote the i-th row of A by A_i, we have $A_iB_\ell = \sum_{j=1}^{n} a_{ij}b_{j\ell}$ for each j, and

$$AB = \begin{bmatrix} A_1B_1 & A_1B_2 & \cdots & A_1B_p \\ A_2B_1 & A_2B_2 & \cdots & A_2B_p \\ \vdots & \vdots & & \vdots \\ A_mB_1 & A_mB_2 & \cdots & A_mB_p \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}b_{j1} & \sum_{j=1}^{n} a_{1j}b_{j2} & \cdots & \sum_{j=1}^{n} a_{1j}b_{jp} \\ \sum_{j=1}^{n} a_{2j}b_{j1} & \sum_{j=1}^{n} a_{2j}b_{j2} & \cdots & \sum_{j=1}^{n} a_{2j}b_{jp} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{n} a_{mj}b_{jp} & \sum_{j=1}^{n} a_{mj}b_{j2} & \cdots & \sum_{j=1}^{n} a_{mj}b_{jp} \end{bmatrix} = \begin{bmatrix} \underline{A_1B} \\ \underline{A_2B} \\ \vdots \\ \overline{A_mB} \end{bmatrix}$$

2. **Examples.**

(a) Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 \\ 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix}$.

Write $A_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \end{bmatrix}$, $A_3 = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \end{bmatrix}$, $A_4 = \begin{bmatrix} 4 & 5 & 6 & 7 & 8 \end{bmatrix}$.

Write $B_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $B_2 = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}$.

We have $A = \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline A_3 \\ \hline A_4 \end{array}\right]$, $B = [\ B_1 \mid B_2\ ]$. Then $AB = \begin{bmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \\ A_3B_1 & A_3B_2 \\ A_4B_1 & A_4B_2 \end{bmatrix} = \begin{bmatrix} 40 & 115 \\ 50 & 150 \\ 60 & 185 \\ 70 & 220 \end{bmatrix}$.

(b) Let $A = \begin{bmatrix} 1 & -1 & 1 & 6 & 1 \\ 6 & 4 & 1 & 4 & -2 \\ 2 & 3 & 2 & -1 & 3 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -5 \\ 2 & -4 & 1 \\ -1 & 1 & 2 \\ 4 & 2 & -3 \\ 6 & 3 & 4 \end{bmatrix}$.

We have $AB = \begin{bmatrix} 28 & 20 & -18 \\ 17 & -13 & -44 \\ 20 & -3 & 12 \\ 10 & -1 & -3 \end{bmatrix}$.

3. **Definition. (Identity matrix.)**

For each positive integer n, the $(n \times n)$-matrix whose (k, k)-th entry is 1 for each k and whose every other entry is 0 is called the identity matrix, and is denoted by I_n.

Remark. $I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$.

4. **Theorem (1). (Basic properties of matrix multiplication.)**

The statements below hold:

(a) *Suppose A is an $(m \times n)$-matrix with real entries. Then $I_m A = A = AI_n$.*

(b) *Suppose A is an $(m \times n)$-matrix and B is an $(n \times p)$-matrix. Suppose α is a real number. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.*

(c) *Suppose A is an $(m \times n)$-matrix and B, C are $(n \times p)$-matrices. Then $A(B + C) = (AB) + (AC)$.*

(d) *Suppose A, B are $(m \times n)$-matrices and C is an $(n \times p)$-matrix. Then $(A + B)C = (AB) + (BC)$.*

Proof. Exercise. (Imitate what is done for verifying the basic properties on addition and scalar multiplication for matrices. In each case verify the corresponding entries on the two sides of an equality are equal to each other. Then apply the definition of matrix equality.)

Remark on terminologies.

(a) Statement (a) is how the 'Law of Existence of Multiplicative Identity' for matrix multiplication. The identity matrices of various sizes are the 'multiplicative identities' concerned.

(b) Statements (c), (d) are collectively known as the 'Distributive Laws' for matrix addition and matrix multiplication.

5. **Lemma (2). (Special case of associativity of matrix multiplication.)**

Suppose A is a row vector with n entries, given by $A = [\ a_1 \quad a_2 \quad \cdots \quad a_n\ ]$, B is an $(n \times p)$-matrix whose (j, k)-th entry is denoted by b_{jk}, and C is a column vectors with p entries, given by $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$.

Then $(AB)C = A(BC) = \displaystyle\sum_{j=1}^{n} \sum_{k=1}^{p} a_j b_{jk} c_k$.

Proof.

- Denote the k-th column of B by $B_{\text{col-}k}$. We have

$$\begin{aligned} AB &= A[\ B_{\text{col-}1} \mid B_{\text{col-}2} \mid \cdots \mid B_{\text{col-}p}\ ] = [\ AB_{\text{col-}1} \mid AB_{\text{col-}2} \mid \cdots \mid AB_{\text{col-}p}\ ] \\ &= \left[\sum_{j=1}^{n} a_j b_{j1} \quad \sum_{j=1}^{n} a_j b_{j2} \quad \cdots \quad \sum_{j=1}^{n} a_j b_{jp} \right] \end{aligned}$$

Then $(AB)C = \left(\displaystyle\sum_{j=1}^{n} a_j b_{j1}\right) c_1 + \left(\displaystyle\sum_{j=1}^{n} a_j b_{j2}\right) c_2 + \cdots + \left(\displaystyle\sum_{j=1}^{n} a_j b_{jp}\right) c_p = \displaystyle\sum_{k=1}^{p} c_k \left(\sum_{j=1}^{n} a_j b_{jk}\right) = \sum_{k=1}^{p} \sum_{j=1}^{n} a_j b_{jk} c_k$.

So (AB)C is the sum of all the $a_j b_{jk} c_k$'s, each copy exactly once.

- Denote the j-th row of B by $B_{\text{row-}j}$. We have

$$BC = \begin{bmatrix} B_{\text{row-}1} \\ \hline B_{\text{row-}2} \\ \hline \vdots \\ \hline B_{\text{row-}n} \end{bmatrix} C = \begin{bmatrix} B_{\text{row-}1}C \\ \hline B_{\text{row-}2}C \\ \hline \vdots \\ \hline B_{\text{row-}n}C \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{p} b_{1k}c_k \\ \sum_{k=1}^{p} b_{2k}c_k \\ \vdots \\ \sum_{k=1}^{p} b_{nk}c_k \end{bmatrix}$$

Then $A(BC) = a_1\left(\sum_{k=1}^{p} b_{1k}c_k\right) + a_2\left(\sum_{k=1}^{p} b_{2k}c_k\right) + \cdots + a_n\left(\sum_{k=1}^{p} b_{nk}c_k\right) = \sum_{j=1}^{n} a_j\left(\sum_{k=1}^{p} b_{jk}\right) = \sum_{j=1}^{n}\sum_{k=1}^{p} a_j b_{jk} c_k.$

So $A(BC)$ is also the sum of all the $a_j b_{jk} c_k$'s, each copy exactly once.

Hence $(AB)C = A(BC)$.

6. **Illustration of the idea in the argument for Lemma (2).**

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$ $C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$.

- We have $B = \begin{bmatrix} B_{\text{col-}1} \mid B_{\text{col-}2} \mid B_{\text{col-}3} \mid B_{\text{col-}4} \end{bmatrix}$, in which $B_{\text{col-}k} = \begin{bmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{bmatrix}$ for each $k = 1, 2, 3, 4$.

  Then, for each $k = 1, 2, 3, 4$, we have $AB_{\text{col-}k} = a_1b_{1k} + a_2b_{2k} + a_3b_{3k}$. This is the k-th entry in the (1×4)-row matrix AB.
  So

$$\begin{aligned} (AB)C &= (a_1b_{11} + a_2b_{21} + a_3b_{31})c_1 \\ &\quad +(a_1b_{12} + a_2b_{22} + a_3b_{32})c_2 \\ &\quad +(a_1b_{13} + a_2b_{23} + a_3b_{33})c_3 \\ &\quad +(a_1b_{14} + a_2b_{24} + a_3b_{34})c_4 \\ &= a_1b_{11}c_1 + a_1b_{12}c_2 + a_1b_{13}c_3 + a_1b_{14}c_4 \\ &\quad +a_2b_{21}c_1 + a_2b_{22}c_2 + a_2b_{23}c_3 + a_2b_{24}c_4 \\ &\quad +a_3b_{31}c_1 + a_3b_{32}c_2 + a_3b_{33}c_3 + a_3b_{34}c_4 \end{aligned}$$

  which is the sum of all the $a_j b_{jk} c_k$'s, each copy exactly once.

- We have $B = \begin{bmatrix} B_{\text{row-}1} \\ \hline B_{\text{row-}2} \\ \hline B_{\text{row-}3} \end{bmatrix}$, in which $B_{\text{row-}j} = \begin{bmatrix} b_{j1} & b_{j2} & b_{j3} & b_{j4} \end{bmatrix}$ for each $j = 1, 2, 3$.

  Then, for each $j = 1, 2, 3$, we have $B_{\text{row-}j}C = b_{j1}c_1 + b_{j2}c_2 + b_{j3}c_3 + b_{j4}c_4$. This is the j-th entry in the (3×1)-column matrix BC.
  So

$$\begin{aligned} A(BC) &= a_1(b_{11}c_1 + b_{12}c_2 + b_{13}c_3 + b_{14}c_4) \\ &\quad +a_2(b_{21}c_1 + b_{22}c_2 + b_{23}c_3 + b_{24}c_4) \\ &\quad +a_3(b_{31}c_1 + b_{32}c_2 + b_{33}c_3 + b_{34}c_4) \\ &= a_1b_{11}c_1 + a_1b_{12}c_2 + a_1b_{13}c_3 + a_1b_{14}c_4 \\ &\quad +a_2b_{21}c_1 + a_2b_{22}c_2 + a_2b_{23}c_3 + a_2b_{24}c_4 \\ &\quad +a_3b_{31}c_1 + a_3b_{32}c_2 + a_3b_{33}c_3 + a_3b_{34}c_4 \end{aligned}$$

  which is the sum of all the $a_j b_{jk} c_k$'s, each copy exactly once.

Hence $(AB)C = A(BC)$ in this particular case indeed.

7. **Theorem (3). (Associativity of matrix multiplication.)**

Suppose A is an $(m \times n)$-matrix, B is an $(n \times p)$-matrix, and C is an $(p \times q)$-matrix. Then $(AB)C = A(BC)$.

Proof.

For each i, denote the i-th row of A by A_i. (So $A = \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{array}\right]$.)

For each ℓ, denote the ℓ-th column of C by C_ℓ. (So $C = [\ C_1 \mid C_2 \mid \cdots \mid C_q\ ]$.)

- We have $AB = \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{array}\right] B = \left[\begin{array}{c} A_1B \\ \hline A_2B \\ \hline \vdots \\ \hline A_mB \end{array}\right]$.

  Then $(AB)C = \left[\begin{array}{c} A_1B \\ \hline A_2B \\ \hline \vdots \\ \hline A_mB \end{array}\right] [\ C_1 \mid C_2 \mid \cdots \mid C_q\ ] = \begin{bmatrix} (A_1B)C_1 & (A_1B)C_2 & \cdots & (A_1B)C_q \\ (A_2B)C_1 & (A_2B)C_2 & \cdots & (A_2B)C_q \\ \vdots & \vdots & & \vdots \\ (A_mB)C_1 & (A_mB)C_2 & \cdots & (A_mB)C_q \end{bmatrix}$.
- We have $BC = B[\ C_1 \mid C_2 \mid \cdots \mid C_q\ ] = [\ BC_1 \mid BC_2 \mid \cdots \mid BC_q\ ]$.

  Then $A(BC) = \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{array}\right] [\ BC_1 \mid BC_2 \mid \cdots \mid BC_q\ ] = \begin{bmatrix} A_1(BC_1) & A_1(BC_2) & \cdots & A_1(BC_q) \\ A_2(BC_1) & A_2(BC_2) & \cdots & A_2(BC_q) \\ \vdots & \vdots & & \vdots \\ A_m(BC_1) & A_m(BC_2) & \cdots & A_m(BC_q) \end{bmatrix}$.

By Lemma (2), the equality $(A_iB)C_\ell = A_i(BC_\ell)$ holds for each i and for each ℓ.

Then $(AB)C = A(BC)$.

8. **Matrix multiplication for 'block matrices', introduced through examples.**

   (a) Let A be an $(m \times n)$-matrix, and $B = [\ B_1 \mid B_2 \mid \cdots \mid B_s\ ]$, in which $B_1, B_2, \cdots, B_s$ be matrices all with n rows and with $p_1, p_2, \cdots, p_s$ columns respectively.

   Then $AB = A[\ B_1 \mid B_2 \mid \cdots \mid B_s\ ] = [\ AB_1 \mid AB_2 \mid \cdots \mid AB_s\ ]$.

   Remark. When $s = p$ and $B_1, B_2, \cdots, B_p$ are the individual columns of B, the above formula reduces to the definition for the product AB.

   (b) Let $A = \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_s \end{array}\right]$, in which $A_1, A_2, \cdots, A_s$ be matrices all with n columns and with $m_1, m_2, \cdots, m_s$ rows respectively, and B be an $(n \times p)$-matrix.

   Then $AB = \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_s \end{array}\right] B = \left[\begin{array}{c} A_1B \\ \hline A_2B \\ \hline \vdots \\ \hline A_sB \end{array}\right]$.

   Remark. When $s = m$ and $A_1, A_2, \cdots, A_m$ are the individual rows of A, the above formula reduces to the definition for the product AB.

   (c) Let $A = [\ A_1 \mid A_2 \mid \cdots \mid A_s\ ]$, in which $A_1, A_2, \cdots, A_s$ are matrices with m rows and with $n_1, n_2, \cdots, n_s$ columns respectively

   Let $B = \left[\begin{array}{c} B_1 \\ \hline B_2 \\ \hline \vdots \\ \hline B_s \end{array}\right]$, in which $B_1, B_2, \cdots, B_s$ are matrices with $n_1, n_2, \cdots, n_s$ rows respectively and with p columns.

   (So for each k, the number of columns of A_k and the number of rows of B_k are the same, and note that A_kB_k is an $(m \times p)$-matrix. Also note that AB is an $(m \times p)$-matrix)

   Then $AB = A_1B_1 + A_2B_2 + \cdots + A_sB_s$.

   Illustration.

   Let $A = \left[\begin{array}{ccc|c|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{array}\right]$, $B = \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ \hline b_{41} & b_{42} \\ \hline b_{51} & b_{52} \\ b_{61} & b_{62} \end{array}\right]$.

   Let $A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $A_2 = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix}$, $A_3 = \begin{bmatrix} a_{15} & a_{16} \\ a_{25} & a_{26} \\ a_{35} & a_{36} \end{bmatrix}$.

Let $B_1 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$, $B_1 = \begin{bmatrix} b_{41} & b_{42} \end{bmatrix}$, $B_3 = \begin{bmatrix} b_{51} & b_{52} \\ b_{61} & b_{62} \end{bmatrix}$.

We have $A = \left[\begin{array}{c|c|c} A_1 & A_2 & A_3 \end{array}\right]$, $B = \left[\begin{array}{c} B_1 \\ \hline B_2 \\ \hline B_3 \end{array}\right]$. Note that

$$A_1B_1 = \begin{bmatrix} \sum_{k=1}^{3} a_{1k}b_{k1} & \sum_{k=1}^{3} a_{1k}b_{k2} \\ \sum_{k=1}^{3} a_{2k}b_{k1} & \sum_{k=1}^{3} a_{2k}b_{k2} \\ \sum_{k=1}^{3} a_{3k}b_{k1} & \sum_{k=1}^{3} a_{3k}b_{k2} \end{bmatrix}, A_2B_2 = \begin{bmatrix} a_{14}b_{41} & a_{14}b_{42} \\ a_{24}b_{41} & a_{24}b_{42} \\ a_{34}b_{41} & a_{34}b_{42} \end{bmatrix}, A_3B_3 = \begin{bmatrix} \sum_{k=5}^{6} a_{1k}b_{k1} & \sum_{k=5}^{6} a_{1k}b_{k2} \\ \sum_{k=5}^{6} a_{2k}b_{k1} & \sum_{k=5}^{6} a_{2k}b_{k2} \\ \sum_{k=5}^{6} a_{3k}b_{k1} & \sum_{k=5}^{6} a_{3k}b_{k2} \end{bmatrix}.$$

Then

$$A_1B_1 + A_2B_2 + A_3B_3 = \begin{bmatrix} \sum_{k=1}^{6} a_{1k}b_{k1} & \sum_{k=1}^{6} a_{1k}b_{k2} \\ \sum_{k=1}^{6} a_{2k}b_{k1} & \sum_{k=1}^{6} a_{2k}b_{k2} \\ \sum_{k=1}^{6} a_{3k}b_{k1} & \sum_{k=1}^{6} a_{3k}b_{k2} \end{bmatrix} = AB$$

indeed.

(d) Let $A_{11}, A_{12}, A_{21}, A_{22}$, $B_{11}, B_{12}, B_{21}, B_{22}$ be matrices. Suppose that

- the number of rows of A_{11}, A_{12} are the same,
- the number of rows of A_{21}, A_{22} are the same,
- the number of columns of B_{11}, B_{21} are the same,
- the number of columns of B_{12}, B_{22} are the same,
- the number of columns of each of A_{11}, A_{21} is the same as the number of rows of each of B_{11}, B_{12},
- the number of columns of each of A_{12}, A_{22} is the same as the number of rows of each of B_{21}, B_{22}.

(So there are integers $m_1, m_2, n_1, n_2, p_1, p_2$ so that A_{11} is an $(m_1 \times n_1)$-matrix, A_{12} is an $(m_1 \times n_2)$-matrix, A_{21} is an $(m_2 \times n_1)$-matrix, A_{22} is an $(m_2 \times n_2)$-matrix, B_{11} is an $(n_1 \times p_1)$-matrix, B_{12} is an $(n_1 \times p_2)$-matrix, B_{21} is an $(n_2 \times p_1)$-matrix, B_{22} is an $(n_2 \times p_2)$-matrix.)

Define $A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$, $B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$.

Then $AB = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right]$.

Illustration.

Let

$$A = \left[\begin{array}{cc|cccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \end{array}\right], \quad B = \left[\begin{array}{cc|ccccc} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} \\ b_{21} & b_{22} & b_{23} & a_{24} & b_{25} & b_{26} & b_{27} \\ \hline b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} & b_{37} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} & b_{47} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} & b_{56} & b_{57} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} & b_{67} \end{array}\right].$$

Write

$$A_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} a_{13} & a_{14} & a_{15} & a_{16} \\ a_{23} & a_{24} & a_{25} & a_{26} \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \\ a_{51} & a_{52} \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} a_{33} & a_{34} & a_{35} & a_{36} \\ a_{43} & a_{44} & a_{45} & a_{46} \\ a_{53} & a_{54} & a_{55} & a_{56} \end{bmatrix},$$

and

$$B_{11} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \qquad B_{12} = \begin{bmatrix} b_{13} & b_{14} & b_{15} & b_{16} & b_{17} \\ b_{23} & a_{24} & b_{25} & b_{26} & b_{27} \end{bmatrix}.$$

$$B_{21} = \begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \\ b_{61} & b_{62} \end{bmatrix}. \qquad B_{22} = \begin{bmatrix} b_{33} & b_{34} & b_{35} & b_{36} & b_{37} \\ b_{43} & b_{44} & b_{45} & b_{46} & b_{47} \\ b_{53} & b_{54} & b_{55} & b_{56} & b_{57} \\ b_{63} & b_{64} & b_{65} & b_{66} & b_{67} \end{bmatrix}.$$

So $A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$, $B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$. We have

$$A_{11}B_{11} + A_{12}B_{21} = \cdots = \begin{bmatrix} \sum_{k=1}^{6} a_{1k}b_{k1} & \sum_{k=1}^{6} a_{1k}b_{k2} \\ \sum_{k=1}^{6} a_{2k}b_{k1} & \sum_{k=1}^{6} a_{2k}b_{k2} \end{bmatrix},$$

$$A_{11}B_{12} + A_{12}B_{22} = \cdots = \begin{bmatrix} \sum_{k=1}^{6} a_{1k}b_{k3} & \sum_{k=1}^{6} a_{1k}b_{k4} & \sum_{k=1}^{6} a_{1k}b_{k5} & \sum_{k=1}^{6} a_{1k}b_{k6} & \sum_{k=1}^{6} a_{1k}b_{k7} \\ \sum_{k=1}^{6} a_{2k}b_{k3} & \sum_{k=1}^{6} a_{2k}b_{k4} & \sum_{k=1}^{6} a_{2k}b_{k5} & \sum_{k=1}^{6} a_{2k}b_{k6} & \sum_{k=1}^{6} a_{2k}b_{k7} \end{bmatrix},$$

$$A_{21}B_{11} + A_{22}B_{21} = \cdots = \begin{bmatrix} \sum_{k=1}^{6} a_{3k}b_{k1} & \sum_{k=1}^{6} a_{3k}b_{k2} \\ \sum_{k=1}^{6} a_{4k}b_{k1} & \sum_{k=1}^{6} a_{4k}b_{k2} \\ \sum_{k=1}^{6} a_{5k}b_{k1} & \sum_{k=1}^{6} a_{5k}b_{k2} \end{bmatrix},$$

$$A_{21}B_{12} + A_{22}B_{22} = \cdots = \begin{bmatrix} \sum_{k=1}^{6} a_{3k}b_{k3} & \sum_{k=1}^{6} a_{3k}b_{k4} & \sum_{k=1}^{6} a_{3k}b_{k5} & \sum_{k=1}^{6} a_{3k}b_{k6} & \sum_{k=1}^{6} a_{3k}b_{k7} \\ \sum_{k=1}^{6} a_{4k}b_{k3} & \sum_{k=1}^{6} a_{4k}b_{k4} & \sum_{k=1}^{6} a_{4k}b_{k5} & \sum_{k=1}^{6} a_{4k}b_{k6} & \sum_{k=1}^{6} a_{4k}b_{k7} \\ \sum_{k=1}^{6} a_{5k}b_{k3} & \sum_{k=1}^{6} a_{5k}b_{k4} & \sum_{k=1}^{6} a_{5k}b_{k5} & \sum_{k=1}^{6} a_{5k}b_{k6} & \sum_{k=1}^{6} a_{5k}b_{k7} \end{bmatrix}.$$

So the equality $AB = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right]$ holds indeed.

The recurrent feature in the manipulation is the fact that the (i, j)-th entry of AB, which is

$$\sum_{k=1}^{6} a_{ik}b_{kj},$$

can be re-written as

$$\sum_{k=1}^{2} a_{ik}b_{kj} + \sum_{k=3}^{6} a_{ik}b_{kj}.$$