1. **Definition. (Matrix Multiplication.)**

(a) *Let A be a row vector with n entries, given by* $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$, *and B be a column vector with n entries, given by* $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

We define the product AB to be the (1×1)*-matrix* $[a_1b_1 + a_2b_2 + \cdots + a_nb_n]$.

For future convenience we abuse notations to confuse as the number $a_1b_1 + a_2b_2 + \cdots + a_nb_n$.

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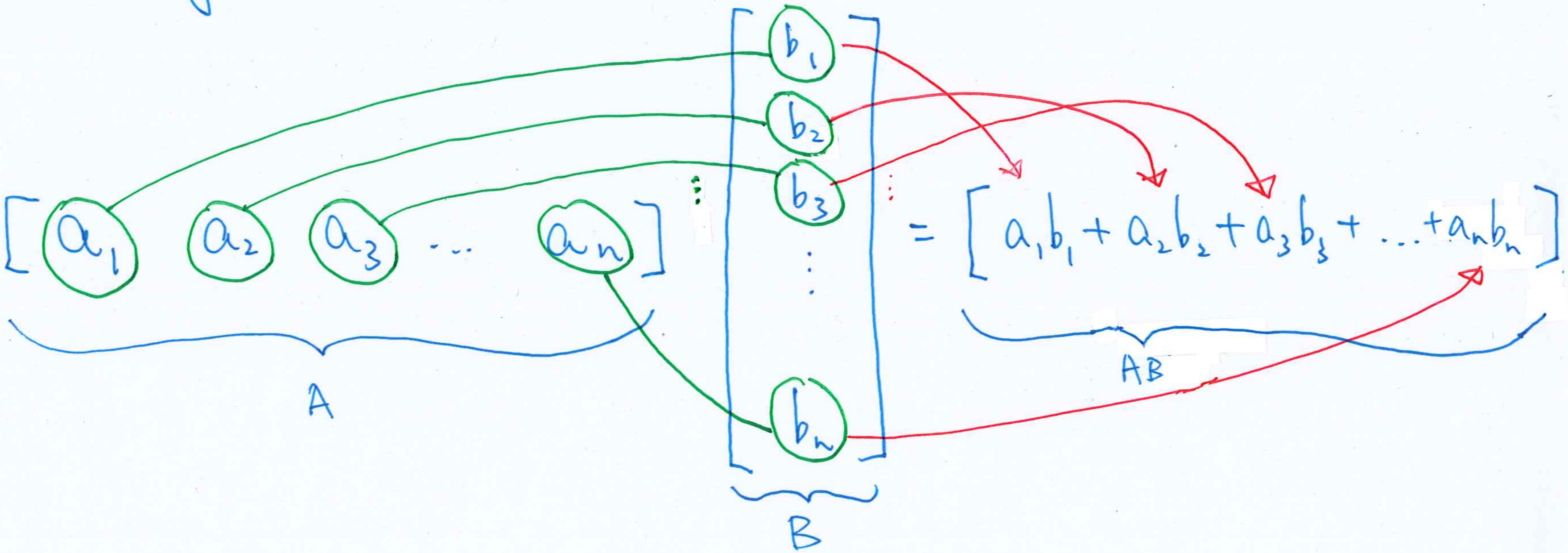
Visualization of the above description:



(b) *Let A be an $(m \times n)$-matrix, whose k-th row is denoted by A_k, and B be a column vector with n entries, given by* $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

We define the product AB to be the column vector with m entries, given by $AB = \begin{bmatrix} A_1B \\ A_2B \\ \vdots \\ A_mB \end{bmatrix}$. *(The k-th entry of AB is A_kB.)*

Remark. Denoting the (i, j)-th entry of A by a_{ij}, the k-th entry of AB is given by the number $\sum_{j=1}^{n} a_{kj}b_j = a_{k1}b_1 + a_{k2}b_2 + \cdots + a_{kn}b_n$.

We have $AB = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + \cdots + a_{2n}b_n \\ \vdots \\ a_{m1}b_1 + a_{m2}b_2 + \cdots + a_{mn}b_n \end{bmatrix}$.

(b) *Let A be an $(m \times n)$-matrix, whose k-th row is denoted by A_k, and B be a column vector with n entries, given by* $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

$$A = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline a_{21} & a_{22} & \cdots & a_{2n} \\ \hline & \vdots & & \\ \hline a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

We define the product AB to be the column vector with m entries, given by $AB = \begin{bmatrix} A_1B \\ A_2B \\ \vdots \\ A_mB \end{bmatrix}$. *(The k-th entry of AB is A_kB.)*

$$A_kB = \begin{bmatrix} a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_{k1}b_1 + a_{k2}b_2 + \cdots + a_{kn}b_n \end{bmatrix}$$

Now think of this (1×1)-matrix as just a number.

Remark. Denoting the (i, j)-th entry of A by a_{ij}, the k-th entry of AB is given by the number $\sum_{j=1}^{n} a_{kj}b_j = a_{k1}b_1 + a_{k2}b_2 + \cdots + a_{kn}b_n$.

We have $AB = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + \cdots + a_{2n}b_n \\ \vdots \\ a_{m1}b_1 + a_{m2}b_2 + \cdots + a_{mn}b_n \end{bmatrix}$.

(c) *Let A be an $(m \times n)$-matrix, and B be an $(n \times p)$-matrix, whose ℓ-th column is denoted by B_ℓ.*
We define the product AB to be the $(m \times p)$-matrix, given by

$$AB = \left[\, AB_1 \,\middle|\, AB_2 \,\middle|\, \cdots \,\middle|\, AB_p \,\right].$$

(The ℓ-th column of AB is AB_ℓ.)

Remark. Denoting the (i, j)-th entry of A by a_{ij}, and the (k, ℓ)-th entry of B by $b_{k\ell}$, the (i, ℓ)-th entry of AB is given by the number $\displaystyle\sum_{j=1}^{n} a_{ij}b_{j\ell} = a_{i1}b_{1\ell} + a_{i2}b_{2\ell} + \cdots + a_{in}b_{n\ell}$.

Denote the i-th row of A by A_i, we have $A_iB_\ell = \displaystyle\sum_{j=1}^{n} a_{ij}b_{j\ell}$ for each j, and

$$AB = \begin{bmatrix} A_1B_1 & A_1B_2 & \cdots & A_1B_p \\ A_2B_1 & A_2B_2 & \cdots & A_2B_p \\ \vdots & \vdots & & \vdots \\ A_mB_1 & A_mB_2 & \cdots & A_mB_p \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}b_{j1} & \sum_{j=1}^{n} a_{1j}b_{j2} & \cdots & \sum_{j=1}^{n} a_{1j}b_{jp} \\ \sum_{j=1}^{n} a_{2j}b_{j1} & \sum_{j=1}^{n} a_{2j}b_{j2} & \cdots & \sum_{j=1}^{n} a_{2j}b_{jp} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{n} a_{mj}b_{jp} & \sum_{j=1}^{n} a_{mj}b_{j2} & \cdots & \sum_{j=1}^{n} a_{mj}b_{jp} \end{bmatrix} = \begin{bmatrix} A_1B \\ \hline A_2B \\ \hline \vdots \\ \hline A_mB \end{bmatrix}$$

$B = [B_1 | B_2 | \cdots | B_p]$ with $B_\ell = \begin{bmatrix} b_{1\ell} \\ b_{2\ell} \\ \vdots \\ b_{n\ell} \end{bmatrix}$

(c) Let A be an $(m \times n)$-matrix, and B be an $(n \times p)$-matrix, whose ℓ-th column is denoted by B_ℓ.

The number of entries in each row of A matches the number of entries in each column of B.

$A = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{bmatrix}$ with $A_k = [a_{k1}\ a_{k2} \cdots a_{kn}]$

We define the product AB to be the $(m \times p)$-matrix, given by

$$AB = \left[\, AB_1 \,\middle|\, AB_2 \,\middle|\, \cdots \,\middle|\, AB_p \,\right].$$

(The ℓ-th column of AB is AB_ℓ.)

$AB_\ell = \begin{bmatrix} A_1 B_\ell \\ A_2 B_\ell \\ \vdots \\ A_m B_\ell \end{bmatrix} = \begin{bmatrix} a_{11}b_{1\ell} + a_{12}b_{2\ell} + \cdots + a_{1n}b_{n\ell} \\ a_{21}b_{1\ell} + a_{22}b_{2\ell} + \cdots + a_{2n}b_{n\ell} \\ \vdots \\ a_{m1}b_{1\ell} + a_{m2}b_{2\ell} + \cdots + a_{mn}b_{n\ell} \end{bmatrix}$

Remark. Denoting the (i, j)-th entry of A by a_{ij}, and the (k, ℓ)-th entry of B by $b_{k\ell}$, the (i, ℓ)-th entry of AB is given by the number $\sum_{j=1}^{n} a_{ij} b_{j\ell} = a_{i1} b_{1\ell} + a_{i2} b_{2\ell} + \cdots + a_{in} b_{n\ell}$.

Denote the i-th row of A by A_i, we have $A_i B_\ell = \sum_{j=1}^{n} a_{ij} b_{j\ell}$ for each j, and

$$AB = \begin{bmatrix} A_1B_1 & A_1B_2 & \cdots & A_1B_p \\ A_2B_1 & A_2B_2 & \cdots & A_2B_p \\ \vdots & \vdots & & \vdots \\ A_mB_1 & A_mB_2 & \cdots & A_mB_p \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}b_{j1} & \sum_{j=1}^{n} a_{1j}b_{j2} & \cdots & \sum_{j=1}^{n} a_{1j}b_{jp} \\ \sum_{j=1}^{n} a_{2j}b_{j1} & \sum_{j=1}^{n} a_{2j}b_{j2} & \cdots & \sum_{j=1}^{n} a_{2j}b_{jp} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{n} a_{mj}b_{jp} & \sum_{j=1}^{n} a_{mj}b_{j2} & \cdots & \sum_{j=1}^{n} a_{mj}b_{jp} \end{bmatrix} = \begin{bmatrix} A_1B \\ \hline A_2B \\ \hline \vdots \\ \hline A_mB \end{bmatrix}$$

$\uparrow$ AB_1 $\uparrow$ AB_2 $\cdots$ $\uparrow$ AB_p

2. **Examples.**

(a) Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 \\ 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix}$.

Write $A_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \end{bmatrix}$, $A_3 = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \end{bmatrix}$, $A_4 = \begin{bmatrix} 4 & 5 & 6 & 7 & 8 \end{bmatrix}$.

Write $B_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $B_2 = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}$.

We have $A = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline A_3 \\ \hline A_4 \end{bmatrix}$, $B = \left[\, B_1 \,\middle|\, B_2 \,\right]$. Then $AB = \begin{bmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \\ A_3B_1 & A_3B_2 \\ A_4B_1 & A_4B_2 \end{bmatrix} = \begin{bmatrix} 40 & 115 \\ 50 & 150 \\ 60 & 185 \\ 70 & 220 \end{bmatrix}$.

(b) Let $A = \begin{bmatrix} 1 & -1 & 1 & 6 & 1 \\ 6 & 4 & 1 & 4 & -2 \\ 2 & 3 & 2 & -1 & 3 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -5 \\ 2 & -4 & 1 \\ -1 & 1 & 2 \\ 4 & 2 & -3 \\ 6 & 3 & 4 \end{bmatrix}$.

We have $AB = \cdots = \begin{bmatrix} 28 & 20 & -18 \\ 17 & -13 & -44 \\ 20 & -3 & 12 \\ 10 & -1 & -3 \end{bmatrix}$.

3. **Definition. (Identity matrix.)**

For each positive integer n, the $(n \times n)$-matrix whose (k, k)-th entry is 1 for each k and whose every other entry is 0 is called the identity matrix, and is denoted by I_n.

Remark. $$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

(b) Let $A = \begin{bmatrix} 1 & -1 & 1 & 6 & 1 \\ 6 & 4 & 1 & 4 & -2 \\ 2 & 3 & 2 & -1 & 3 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -5 \\ 2 & -4 & 1 \\ -1 & 1 & 2 \\ 4 & 2 & -3 \\ 6 & 3 & 4 \end{bmatrix}$.

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Remark. $$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

$I_1 = [1]$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\vdots$

4. **Theorem (1). (Basic properties of matrix multiplication.)**

The statements below hold:

(a) *Suppose* A *is an* $(m \times n)$*-matrix with real entries.*
Then $I_m A = A = A I_n$.

(b) *Suppose* A *is an* $(m \times n)$*-matrix and* B *is an* $(n \times p)$*-matrix. Suppose* α *is a real number.*
Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

(c) *Suppose* A *is an* $(m \times n)$*-matrix and* B, C *are* $(n \times p)$*-matrices.*
Then $A(B + C) = (AB) + (AC)$.

(d) *Suppose* A, B *are* $(m \times n)$*-matrices and* C *is an* $(n \times p)$*-matrix.*
Then $(A + B)C = (AB) + (BC)$.

Remark on terminologies.

(a) Statement (a) is how the 'Law of Existence of Multiplicative Identity' for matrix multiplication. The identity matrices of various sizes are the 'multiplicative identities' concerned.

(b) Statements (c), (d) are collectively known as the 'Distributive Laws' for matrix addition and matrix multiplication.

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The statements below hold:

(a) *Suppose A is an $(m \times n)$-matrix with real entries. Then $I_m A = A = A I_n$.*

Illustration: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) *Suppose A is an $(m \times n)$-matrix and B is an $(n \times p)$-matrix. Suppose α is a real number. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.*

(c) *Suppose A is an $(m \times n)$-matrix and B, C are $(n \times p)$-matrices. Then $A(B + C) = (AB) + (AC)$.*

(d) *Suppose A, B are $(m \times n)$-matrices and C is an $(n \times p)$-matrix. Then $(A + B)C = (AB) + (BC)$.*

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5. **Lemma (2). (Special case of associativity of matrix multiplication.)**

Suppose A is a row vector with n entries, given by

$$A = \left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_n \end{array}\right],$$

B is an $(n \times p)$-matrix whose (j,k)-th entry is denoted by b_{jk}, and
C is a column vectors with p entries, given by

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}.$$

Then

$$(AB)C = A(BC) = \sum_{j=1}^{n} \sum_{k=1}^{p} a_j b_{jk} c_k.$$

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Then

$$(AB)C = A(BC) = \sum_{j=1}^{n} \sum_{k=1}^{p} a_j b_{jk} c_k.$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$ } n rows, p columns

What is it?
It is the sum of all possible np expressions of the form

$$a_j b_{jk} c_k,$$

each expression appearing exactly once.

Interpretation of this equality: whether you multiply A to B first, or multiply B to C first, you will end up with the same product as long as the order A, B, C in this multiplication is maintained.

Proof.

- Denote the k-th column of B by $B_{\text{col-}k}$. We have

$$AB = A\left[\, B_{\text{col-1}} \,\middle|\, B_{\text{col-2}} \,\middle|\, \cdots \,\middle|\, B_{\text{col-}p} \,\right] = \left[\, AB_{\text{col-1}} \,\middle|\, AB_{\text{col-2}} \,\middle|\, \cdots \,\middle|\, AB_{\text{col-}p} \,\right]$$
$$= \left[\sum_{j=1}^{n} a_j b_{j1} \;\; \sum_{j=1}^{n} a_j b_{j2} \;\; \cdots \;\; \sum_{j=1}^{n} a_j b_{jp} \right]$$

Then

$$(AB)C = \left(\sum_{j=1}^{n} a_j b_{j1}\right) c_1 + \left(\sum_{j=1}^{n} a_j b_{j2}\right) c_2 + \cdots + \left(\sum_{j=1}^{n} a_j b_{jp}\right) c_p$$
$$= \sum_{k=1}^{p} c_k \left(\sum_{j=1}^{n} a_j b_{jk}\right) = \sum_{k=1}^{p} \sum_{j=1}^{n} a_j b_{jk} c_k$$

So (AB)C is the sum of all the $a_j b_{jk} c_k$'s, each copy exactly once.

- Denote the j-th row of B by $B_{\text{row-}j}$. We have

$$BC = \left[\begin{array}{c} B_{\text{row-1}} \\ \hline B_{\text{row-2}} \\ \hline \vdots \\ \hline B_{\text{row-}n} \end{array}\right] C = \left[\begin{array}{c} B_{\text{row-1}}C \\ \hline B_{\text{row-2}}C \\ \hline \vdots \\ \hline B_{\text{row-}n}C \end{array}\right] = \begin{bmatrix} \sum\limits_{k=1}^{p} b_{1k}c_k \\ \sum\limits_{k=1}^{p} b_{2k}c_k \\ \vdots \\ \sum\limits_{k=1}^{p} b_{nk}c_k \end{bmatrix}$$

Then

$$\begin{aligned} A(BC) &= a_1\left(\sum_{k=1}^{p} b_{1k}c_k\right) + a_2\left(\sum_{k=1}^{p} b_{2k}c_k\right) + \cdots + a_n\left(\sum_{k=1}^{p} b_{nk}c_k\right) \\ &= \sum_{j=1}^{n} a_j\left(\sum_{k=1}^{p} b_{jk}\right) = \sum_{j=1}^{n}\sum_{k=1}^{p} a_j b_{jk} c_k. \end{aligned}$$

So $A(BC)$ is also the sum of all the $a_j b_{jk} c_k$'s, each copy exactly once.

Hence $(AB)C = A(BC)$.

6. **Illustration of the idea in the argument for Lemma (2).**

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$ $C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$.

- We have $B = \left[\, B_{\text{col-1}} \,\middle|\, B_{\text{col-2}} \,\middle|\, B_{\text{col-3}} \,\middle|\, B_{\text{col-4}} \,\right]$, in which $B_{\text{col-}k} = \begin{bmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{bmatrix}$ for each $k = 1, 2, 3, 4$.

  Then, for each $k = 1, 2, 3, 4$, we have $AB_{\text{col-}k} = a_1 b_{1k} + a_2 b_{2k} + a_3 b_{3k}$. This is the k-th entry in the (1×4)-row matrix AB. So

$$\begin{aligned} (AB)C &= (a_1 b_{11} + a_2 b_{21} + a_3 b_{31}) c_1 \\ &\quad + (a_1 b_{12} + a_2 b_{22} + a_3 b_{32}) c_2 \\ &\quad + (a_1 b_{13} + a_2 b_{23} + a_3 b_{33}) c_3 \\ &\quad + (a_1 b_{14} + a_2 b_{24} + a_3 b_{34}) c_4 \\ &= a_1 b_{11} c_1 + a_1 b_{12} c_2 + a_1 b_{13} c_3 + a_1 b_{14} c_4 \\ &\quad + a_2 b_{21} c_1 + a_2 b_{22} c_2 + a_2 b_{23} c_3 + a_2 b_{24} c_4 \\ &\quad + a_3 b_{31} c_1 + a_3 b_{32} c_2 + a_3 b_{33} c_3 + a_3 b_{34} c_4 \end{aligned}$$

  which is the sum of all the $a_j b_{jk} c_k$'s, each copy exactly once.

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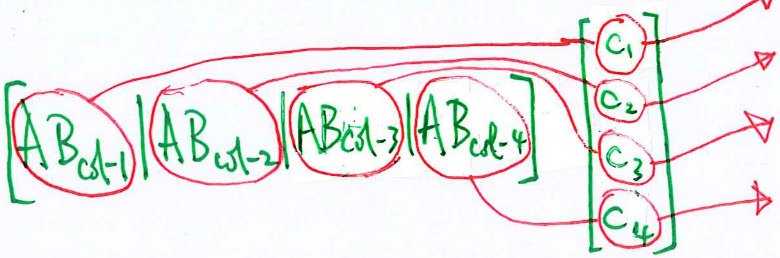
Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$ $C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$.

- We have $B = \left[\, B_{\text{col-1}} \,\middle|\, B_{\text{col-2}} \,\middle|\, B_{\text{col-3}} \,\middle|\, B_{\text{col-4}} \,\right]$, in which $B_{\text{col-}k} = \begin{bmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{bmatrix}$ for each $k = 1, 2, 3, 4$.

  Then, for each $k = 1, 2, 3, 4$, we have $AB_{\text{col-}k} = a_1 b_{1k} + a_2 b_{2k} + a_3 b_{3k}$. This is the k-th entry in the (1×4)-row matrix AB. So



$$
\begin{aligned}
(AB)C &= (a_1 b_{11} + a_2 b_{21} + a_3 b_{31})c_1 && \leftarrow \text{This is } (AB_{\text{col-1}})c_1 \\
&\quad +(a_1 b_{12} + a_2 b_{22} + a_3 b_{32})c_2 && \leftarrow \text{This is } (AB_{\text{col-2}})c_2 \\
&\quad +(a_1 b_{13} + a_2 b_{23} + a_3 b_{33})c_3 && \leftarrow \text{This is } (AB_{\text{col-3}})c_3 \\
&\quad +(a_1 b_{14} + a_2 b_{24} + a_3 b_{34})c_4 && \leftarrow \text{This is } (AB_{\text{col-4}})c_4 \\
&= a_1 b_{11} c_1 + a_1 b_{12} c_2 + a_1 b_{13} c_3 + a_1 b_{14} c_4 \\
&\quad +a_2 b_{21} c_1 + a_2 b_{22} c_2 + a_2 b_{23} c_3 + a_2 b_{24} c_4 \\
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\end{aligned}
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  which is the sum of all the $a_j b_{jk} c_k$'s, each copy exactly once.

- We have $B = \left[\begin{array}{c} B_{\text{row-1}} \\ \hline B_{\text{row-2}} \\ \hline B_{\text{row-3}} \end{array}\right]$, in which $B_{\text{row-}j} = \left[\begin{array}{cccc} b_{j1} & b_{j2} & b_{j3} & b_{j4} \end{array}\right]$ for each $j = 1, 2, 3$.

  Then, for each $j = 1, 2, 3$, we have $B_{\text{row-}j}C = b_{j1}c_1 + b_{j2}c_2 + b_{j3}c_3 + b_{j4}c_4$. This is the j-th entry in the (3×1)-column matrix BC. So

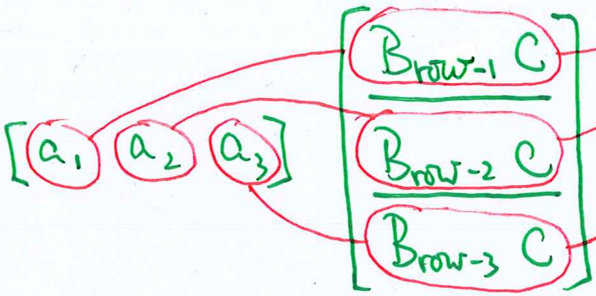
$$\begin{aligned} A(BC) &= a_1(b_{11}c_1 + b_{12}c_2 + b_{13}c_3 + b_{14}c_4) \\ &\quad + a_2(b_{21}c_1 + b_{22}c_2 + b_{23}c_3 + b_{24}c_4) \\ &\quad + a_3(b_{31}c_1 + b_{32}c_2 + b_{33}c_3 + b_{34}c_4) \\ &= a_1b_{11}c_1 + a_1b_{12}c_2 + a_1b_{13}c_3 + a_1b_{14}c_4 \\ &\quad + a_2b_{21}c_1 + a_2b_{22}c_2 + a_2b_{23}c_3 + a_2b_{24}c_4 \\ &\quad + a_3b_{31}c_1 + a_3b_{32}c_2 + a_3b_{33}c_3 + a_3b_{34}c_4 \end{aligned}$$

  which is the sum of all the $a_jb_{jk}c_k$'s, each copy exactly once.

Hence $(AB)C = A(BC)$ in this particular case indeed.

- We have $B = \begin{bmatrix} B_{\text{row-1}} \\ \hline B_{\text{row-2}} \\ \hline B_{\text{row-3}} \end{bmatrix}$, in which $B_{\text{row-}j} = \begin{bmatrix} b_{j1} & b_{j2} & b_{j3} & b_{j4} \end{bmatrix}$ for each $j = 1, 2, 3$.

  Then, for each $j = 1, 2, 3$, we have $B_{\text{row-}j}C = b_{j1}c_1 + b_{j2}c_2 + b_{j3}c_3 + b_{j4}c_4$. This is the j-th entry in the (3×1)-column matrix BC. So



$$
\begin{aligned}
A(BC) &= a_1(b_{11}c_1 + b_{12}c_2 + b_{13}c_3 + b_{14}c_4) \\
&\quad +a_2(b_{21}c_1 + b_{22}c_2 + b_{23}c_3 + b_{24}c_4) \\
&\quad +a_3(b_{31}c_1 + b_{32}c_2 + b_{33}c_3 + b_{34}c_4) \\
&= a_1b_{11}c_1 + a_1b_{12}c_2 + a_1b_{13}c_3 + a_1b_{14}c_4 \\
&\quad +a_2b_{21}c_1 + a_2b_{22}c_2 + a_2b_{23}c_3 + a_2b_{24}c_4 \\
&\quad +a_3b_{31}c_1 + a_3b_{32}c_2 + a_3b_{33}c_3 + a_3b_{34}c_4
\end{aligned}
$$

← This is a_1 (Brow-1 C)
← This is a_2 (Brow-2 C)
← This is a_3 (Brow-3 C)

  which is the sum of all the $a_jb_{jk}c_k$'s, each copy exactly once.

Hence $(AB)C = A(BC)$ in this particular case indeed.

7. **Theorem (3). (Associativity of matrix multiplication.)**

Suppose A is an $(m \times n)$-matrix, B is an $(n \times p)$-matrix, and C is an $(p \times q)$-matrix. Then $(AB)C = A(BC)$.

Proof.

For each i, denote the i-th row of A by A_i. (So $A = \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{array}\right]$.)

For each ℓ, denote the ℓ-th column of C by C_ℓ. (So $C = \left[\begin{array}{c|c|c|c} C_1 & C_2 & \cdots & C_q \end{array}\right]$.)

- We have $AB = \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{array}\right] B = \left[\begin{array}{c} A_1B \\ \hline A_2B \\ \hline \vdots \\ \hline A_mB \end{array}\right]$.

  Then

$$(AB)C = \left[\begin{array}{c} A_1B \\ \hline A_2B \\ \hline \vdots \\ \hline A_mB \end{array}\right] \left[\begin{array}{c|c|c|c} C_1 & C_2 & \cdots & C_q \end{array}\right] = \begin{bmatrix} (A_1B)C_1 & (A_1B)C_2 & \cdots & (A_1B)C_q \\ (A_2B)C_1 & (A_2B)C_2 & \cdots & (A_2B)C_q \\ \vdots & \vdots & & \vdots \\ (A_mB)C_1 & (A_mB)C_2 & \cdots & (A_mB)C_q \end{bmatrix}.$$

7. **Theorem (3). (Associativity of matrix multiplication.)**
Suppose A is an $(m \times n)$-matrix, B is an $(n \times p)$-matrix, and C is an $(p \times q)$-matrix. Then $(AB)C = A(BC)$.

Proof.

For each i, denote the i-th row of A by A_i. (So $A = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{bmatrix}$.)

For each ℓ, denote the ℓ-th column of C by C_ℓ. (So $C = \left[\, C_1 \,|\, C_2 \,|\, \cdots \,|\, C_q \,\right]$.)

- We have $AB = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{bmatrix} B = \begin{bmatrix} A_1B \\ \hline A_2B \\ \hline \vdots \\ \hline A_mB \end{bmatrix}$.

  Then

$$(AB)C = \begin{bmatrix} A_1B \\ \hline A_2B \\ \hline \vdots \\ \hline A_mB \end{bmatrix} \left[\, C_1 \,|\, C_2 \,|\, \cdots \,|\, C_q \,\right] = \begin{bmatrix} (A_1B)C_1 & (A_1B)C_2 & \cdots & (A_1B)C_q \\ (A_2B)C_1 & (A_2B)C_2 & \cdots & (A_2B)C_q \\ \vdots & \vdots & & \vdots \\ (A_mB)C_1 & (A_mB)C_2 & \cdots & (A_mB)C_q \end{bmatrix}.$$

The (i,ℓ)-th entry is $(A_iB)C_\ell$.

The i-th row is A_iB.

The ℓ-th column is C_ℓ.

- We have $BC = B\left[\, C_1 \,\middle|\, C_2 \,\middle|\, \cdots \,\middle|\, C_q \,\right] = \left[\, BC_1 \,\middle|\, BC_2 \,\middle|\, \cdots \,\middle|\, BC_q \,\right]$.

  Then

$$A(BC) = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{bmatrix} \left[\, BC_1 \,\middle|\, BC_2 \,\middle|\, \cdots \,\middle|\, BC_q \,\right] = \begin{bmatrix} A_1(BC_1) & A_1(BC_2) & \cdots & A_1(BC_q) \\ A_2(BC_1) & A_2(BC_2) & \cdots & A_2(BC_q) \\ \vdots & \vdots & & \vdots \\ A_m(BC_1) & A_m(BC_2) & \cdots & A_m(BC_q) \end{bmatrix}.$$

By Lemma (2), the equality $(A_iB)C_\ell = A_i(BC_\ell)$ holds for each i and for each ℓ.

Then $(AB)C = A(BC)$.

- We have $BC = B[\, C_1 \,|\, C_2 \,|\cdots|\, C_q \,] = [\, BC_1 \,|\, BC_2 \,|\cdots|\, BC_q \,]$.
  Then

The (i,ℓ)-th entry is $A_i(BC_\ell)$.

$$A(BC) = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{bmatrix} [\, BC_1 \,|\, BC_2 \,|\cdots|\, BC_q \,] = \begin{bmatrix} A_1(BC_1) & A_1(BC_2) & \cdots & A_1(BC_q) \\ A_2(BC_1) & A_2(BC_2) & \cdots & A_2(BC_q) \\ \vdots & \vdots & & \vdots \\ A_m(BC_1) & A_m(BC_2) & \cdots & A_m(BC_q) \end{bmatrix}.$$

The i-th row is A_i. The ℓ-th column is BC_ℓ.

By Lemma (2), the equality $(A_iB)C_\ell = A_i(BC_\ell)$ holds for each i and for each ℓ.
Then $(AB)C = A(BC)$.

8. **Matrix multiplication for 'block matrices', introduced through examples.**

(a) Let A be an $(m \times n)$-matrix, and $B = \left[\, B_1 \,\middle|\, B_2 \,\middle|\, \cdots \,\middle|\, B_s \,\right]$, in which $B_1, B_2, \cdots, B_s$ be matrices all with n rows and with $p_1, p_2, \cdots, p_s$ columns respectively.
Then $AB = A\left[\, B_1 \,\middle|\, B_2 \,\middle|\, \cdots \,\middle|\, B_s \,\right] = \left[\, AB_1 \,\middle|\, AB_2 \,\middle|\, \cdots \,\middle|\, AB_s \,\right]$.

Remark. When $s = p$ and $B_1, B_2, \cdots, B_p$ are the individual columns of B, the above formula reduces to the definition for the product AB.

(b) Let $A = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_s \end{bmatrix}$, in which $A_1, A_2, \cdots, A_s$ be matrices all with n columns and with $m_1, m_2, \cdots, m_s$ rows respectively, and B be an $(n \times p)$-matrix.

Then $AB = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_s \end{bmatrix} B = \begin{bmatrix} A_1B \\ \hline A_2B \\ \hline \vdots \\ \hline A_sB \end{bmatrix}$.

Remark. When $s = m$ and $A_1, A_2, \cdots, A_m$ are the individual rows of A, the above formula reduces to the definition for the product AB.

(c) Let $A = \left[\, A_1 \,\middle|\, A_2 \,\middle|\, \cdots \,\middle|\, A_s \,\right]$, in which $A_1, A_2, \cdots, A_s$ are matrices with m rows and with $n_1, n_2, \cdots, n_s$ columns respectively

Let $B = \begin{bmatrix} B_1 \\ \hline B_2 \\ \hline \vdots \\ \hline B_s \end{bmatrix}$, in which $B_1, B_2, \cdots, B_s$ are matrices with $n_1, n_2, \cdots, n_s$ rows respectively and with p columns.

(So for each k, the number of columns of A_k and the number of rows of B_k are the same, and note that A_kB_k is an $(m \times p)$-matrix. Also note that AB is an $(m \times p)$-matrix)

Then $AB = A_1B_1 + A_2B_2 + \cdots + A_sB_s$.

Illustration.

Let $A = \left[\begin{array}{ccc|c|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{array}\right]$, $B = \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ \hline b_{41} & b_{42} \\ \hline b_{51} & b_{52} \\ b_{61} & b_{62} \end{array}\right]$.

Let $A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $A_2 = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix}$, $A_3 = \begin{bmatrix} a_{15} & a_{16} \\ a_{25} & a_{26} \\ a_{35} & a_{36} \end{bmatrix}$.

Let $B_1 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$, $B_1 = \begin{bmatrix} b_{41} & b_{42} \end{bmatrix}$, $B_3 = \begin{bmatrix} b_{51} & b_{52} \\ b_{61} & b_{62} \end{bmatrix}$.

We have $A = \left[A_1 \middle| A_2 \middle| A_3 \right]$, $B = \begin{bmatrix} B_1 \\ \hline B_2 \\ \hline B_3 \end{bmatrix}$. Note that

$$A_1B_1 = \begin{bmatrix} \sum_{k=1}^{3} a_{1k}b_{k1} & \sum_{k=1}^{3} a_{1k}b_{k2} \\ \sum_{k=1}^{3} a_{2k}b_{k1} & \sum_{k=1}^{3} a_{2k}b_{k2} \\ \sum_{k=1}^{3} a_{3k}b_{k1} & \sum_{k=1}^{3} a_{3k}b_{k2} \end{bmatrix}, A_2B_2 = \begin{bmatrix} a_{14}b_{41} & a_{14}b_{42} \\ a_{24}b_{41} & a_{24}b_{42} \\ a_{34}b_{41} & a_{34}b_{42} \end{bmatrix}, A_3B_3 = \begin{bmatrix} \sum_{k=5}^{6} a_{1k}b_{k1} & \sum_{k=5}^{6} a_{1k}b_{k2} \\ \sum_{k=5}^{6} a_{2k}b_{k1} & \sum_{k=5}^{6} a_{2k}b_{k2} \\ \sum_{k=5}^{6} a_{3k}b_{k1} & \sum_{k=5}^{6} a_{3k}b_{k2} \end{bmatrix}.$$

Then

$$A_1B_1 + A_2B_2 + A_3B_3 = \begin{bmatrix} \sum_{k=1}^{6} a_{1k}b_{k1} & \sum_{k=1}^{6} a_{1k}b_{k2} \\ \sum_{k=1}^{6} a_{2k}b_{k1} & \sum_{k=1}^{6} a_{2k}b_{k2} \\ \sum_{k=1}^{6} a_{3k}b_{k1} & \sum_{k=1}^{6} a_{3k}b_{k2} \end{bmatrix} = AB$$

indeed.

(d) Let $A_{11}, A_{12}, A_{21}, A_{22}$, $B_{11}, B_{12}, B_{21}, B_{22}$ be matrices. Suppose that

- the number of rows of A_{11}, A_{12} are the same,
- the number of rows of A_{21}, A_{22} are the same,
- the number of columns of B_{11}, B_{21} are the same,
- the number of columns of B_{12}, B_{22} are the same,
- the number of columns of each of A_{11}, A_{21} is the same as the number of rows of each of B_{11}, B_{12},
- the number of columns of each of A_{12}, A_{22} is the same as the number of rows of each of B_{21}, B_{22}.

(So there are integers $m_1, m_2, n_1, n_2, p_1, p_2$ so that A_{11} is an $(m_1 \times n_1)$-matrix, A_{12} is an $(m_1 \times n_2)$-matrix, A_{21} is an $(m_2 \times n_1)$-matrix, A_{22} is an $(m_2 \times n_2)$-matrix, B_{11} is an $(n_1 \times p_1)$-matrix, B_{12} is an $(n_1 \times p_2)$-matrix, B_{21} is an $(n_2 \times p_1)$-matrix, B_{22} is an $(n_2 \times p_2)$-matrix.)

Define $A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$, $B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$.

Then $AB = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right]$.

Illustration.

Let

$$A = \left[\begin{array}{cc|cccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \end{array}\right], \quad B = \left[\begin{array}{cc|ccccc} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} \\ b_{21} & b_{22} & b_{23} & a_{24} & b_{25} & b_{26} & b_{27} \\ \hline b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} & b_{37} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} & b_{47} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} & b_{56} & b_{57} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} & b_{67} \end{array}\right].$$

Write

$$A_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} a_{13} & a_{14} & a_{15} & a_{16} \\ a_{23} & a_{24} & a_{25} & a_{26} \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \\ a_{51} & a_{52} \end{bmatrix}, \quad A_{22} = \begin{bmatrix} a_{33} & a_{34} & a_{35} & a_{36} \\ a_{43} & a_{44} & a_{45} & a_{46} \\ a_{53} & a_{54} & a_{55} & a_{56} \end{bmatrix},$$

and

$$B_{11} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad B_{12} = \begin{bmatrix} b_{13} & b_{14} & b_{15} & b_{16} & b_{17} \\ b_{23} & a_{24} & b_{25} & b_{26} & b_{27} \end{bmatrix}.$$

$$B_{21} = \begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \\ b_{61} & b_{62} \end{bmatrix}. \quad B_{22} = \begin{bmatrix} b_{33} & b_{34} & b_{35} & b_{36} & b_{37} \\ b_{43} & b_{44} & b_{45} & b_{46} & b_{47} \\ b_{53} & b_{54} & b_{55} & b_{56} & b_{57} \\ b_{63} & b_{64} & b_{65} & b_{66} & b_{67} \end{bmatrix}.$$

So $A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$, $B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$. We have

$$A_{11}B_{11} + A_{12}B_{21} = \cdots = \begin{bmatrix} \sum_{k=1}^{6} a_{1k}b_{k1} & \sum_{k=1}^{6} a_{1k}b_{k2} \\ \sum_{k=1}^{6} a_{2k}b_{k1} & \sum_{k=1}^{6} a_{2k}b_{k2} \end{bmatrix},$$

$$A_{11}B_{12} + A_{12}B_{22} = \cdots = \begin{bmatrix} \sum_{k=1}^{6} a_{1k}b_{k3} & \sum_{k=1}^{6} a_{1k}b_{k4} & \sum_{k=1}^{6} a_{1k}b_{k5} & \sum_{k=1}^{6} a_{1k}b_{k6} & \sum_{k=1}^{6} a_{1k}b_{k7} \\ \sum_{k=1}^{6} a_{2k}b_{k3} & \sum_{k=1}^{6} a_{2k}b_{k4} & \sum_{k=1}^{6} a_{2k}b_{k5} & \sum_{k=1}^{6} a_{2k}b_{k6} & \sum_{k=1}^{6} a_{2k}b_{k7} \end{bmatrix},$$

$$A_{21}B_{11} + A_{22}B_{21} = \cdots = \begin{bmatrix} \sum_{k=1}^{6} a_{3k}b_{k1} & \sum_{k=1}^{6} a_{3k}b_{k2} \\ \sum_{k=1}^{6} a_{4k}b_{k1} & \sum_{k=1}^{6} a_{4k}b_{k2} \\ \sum_{k=1}^{6} a_{5k}b_{k1} & \sum_{k=1}^{6} a_{5k}b_{k2} \end{bmatrix},$$

$$A_{21}B_{12} + A_{22}B_{22} = \cdots = \begin{bmatrix} \sum_{k=1}^{6} a_{3k}b_{k3} & \sum_{k=1}^{6} a_{3k}b_{k4} & \sum_{k=1}^{6} a_{3k}b_{k5} & \sum_{k=1}^{6} a_{3k}b_{k6} & \sum_{k=1}^{6} a_{3k}b_{k7} \\ \sum_{k=1}^{6} a_{4k}b_{k3} & \sum_{k=1}^{6} a_{4k}b_{k4} & \sum_{k=1}^{6} a_{4k}b_{k5} & \sum_{k=1}^{6} a_{4k}b_{k6} & \sum_{k=1}^{6} a_{4k}b_{k7} \\ \sum_{k=1}^{6} a_{5k}b_{k3} & \sum_{k=1}^{6} a_{5k}b_{k4} & \sum_{k=1}^{6} a_{5k}b_{k5} & \sum_{k=1}^{6} a_{5k}b_{k6} & \sum_{k=1}^{6} a_{5k}b_{k7} \end{bmatrix}.$$

So the equality $AB = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right]$ holds indeed.

The recurrent feature in the manipulation is the fact that the (i, j)-th entry of AB, which is

$$\sum_{k=1}^{6} a_{ik}b_{kj},$$

can be re-written as

$$\sum_{k=1}^{2} a_{ik}b_{kj} + \sum_{k=3}^{6} a_{ik}b_{kj}.$$