1. **Definition. (Augmented matrix representation.)**

   Consider the system of m linear equations with n unknowns

$$(S): \left\{ \begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \right.$$

   in which the a_{ij}'s, b_i's are the given's and the x_j's are the unknowns.

   The $(m \times (n+1))$-matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

   is called the augmented matrix representation of the system (S).

2. **Question.** Why are we interested in 'augmented matrix representations'?

   Answer. Augmented matrix representations and row operations can be used as short-hand for presenting the manipulations for solving systems of linear equations.

   Illustration.

   (a) Inspect how we apply 'equation operations' to manipulate a given system of equations, say, (S)

$$(S): \left\{ \begin{array}{ccccccccc} x_1 & + & 2x_2 & & & + & x_4 & = & 7 \\ x_1 & + & x_2 & + & x_3 & - & x_4 & = & 3 \\ 3x_1 & + & x_2 & + & 5x_3 & - & 7x_4 & = & 1 \end{array} \right.$$

   towards finding all solutions of the system. (We start by re-labeling (S) as (S_1).)

$$(S_1) \quad \left\{ \begin{array}{ccccccccccc} 1x_1 & + & 2x_2 & + & 0x_3 & + & 1x_4 & = & 7 & — & ① \\ 1x_1 & + & 1x_2 & + & 1x_3 & + & (-1)x_4 & = & 3 & — & ② \\ 3x_1 & + & 1x_2 & + & 5x_3 & + & (-7)x_4 & = & 1 & — & ③ \end{array} \right.$$

$$(-1) \times ① + ② : \quad (S_2) \quad \left\{ \begin{array}{ccccccccccc} 1x_1 & + & 2x_2 & + & 0x_3 & + & 1x_4 & = & 7 & — & ① \\ 0x_1 & + & (-1)x_2 & + & 1x_3 & + & (-2)x_4 & = & -4 & — & ④ \\ 3x_1 & + & 1x_2 & + & 5x_3 & + & (-7)x_4 & = & 1 & — & ③ \end{array} \right.$$

$$(-3) \times ① + ③ : \quad (S_3) \quad \left\{ \begin{array}{ccccccccccc} 1x_1 & + & 2x_2 & + & 0x_3 & + & 1x_4 & = & 7 & — & ① \\ 0x_1 & + & (-1)x_2 & + & 1x_3 & + & (-2)x_4 & = & -4 & — & ④ \\ 0x_1 & + & (-5)x_2 & + & 5x_3 & + & (-10)x_4 & = & -20 & — & ⑤ \end{array} \right.$$

$$(-1) \times ④ : \quad (S_4) \quad \left\{ \begin{array}{ccccccccccc} 1x_1 & + & 2x_2 & + & 0x_3 & + & 1x_4 & = & 7 & — & ① \\ 0x_1 & + & 1x_2 & + & (-1)x_3 & + & 2x_4 & = & 4 & — & ⑥ \\ 0x_1 & + & (-5)x_2 & + & 5x_3 & + & (-10)x_4 & = & -20 & — & ⑤ \end{array} \right.$$

$$5 \times ⑥ + ⑤ : \quad (S_5) \quad \left\{ \begin{array}{ccccccccccc} 1x_1 & + & 2x_2 & + & 0x_3 & + & 1x_4 & = & 7 & — & ① \\ 0x_1 & + & 1x_2 & + & (-1)x_3 & + & 2x_4 & = & 4 & — & ⑥ \\ 0x_1 & + & 0x_2 & + & 0x_3 & + & 0x_4 & = & 0 & — & ⑦ \end{array} \right.$$

$$(-2) \times ⑥ + ① : \quad (S_6) \quad \left\{ \begin{array}{ccccccccccc} 1x_1 & + & 0x_2 & + & 2x_3 & + & (-3)x_4 & = & -1 & — & ① \\ 0x_1 & + & 1x_2 & + & (-1)x_3 & + & 2x_4 & = & 4 & — & ⑥ \\ 0x_1 & + & 0x_2 & + & 0x_3 & + & 0x_4 & = & 0 & — & ⑦ \end{array} \right.$$

   Hence the solutions of (S) is described by $(x_1, x_2, x_3, x_4) = (-1 - 2s + 3t, 4 + s - 2t, s, t)$ where s, t are arbitrary numbers.

   (b) We observe:—

   It is the 'effects' on the 'givens' in the successive systems resultant from the applications of 'equations operations' that matter.

   We may think of the symbols '+', '=' and 'x_1', ..., 'x_4' as 'hangers' for the 'givens'.

   If we can find a way to keep track of the 'givens' in the successive systems, we can do without writing the symbols '+', '=' and 'x_1', ..., 'x_4' repeatedly.

   The augmented matrix representation is a tool for this purpose.

(c) Just write down the sequence of augmented matrix representations $(C_1), ..., (C_6)$ of the system of systems $(S_1), ..., (S_6)$ in the manipulations.

Ask: How are these matrices related to each other?

Answer. The sequence of 'equations operations' linking $(S_1), ..., (S_6)$ translates as the sequence of row operations linking $C_1, ..., C_6$:

$$C = C_1 = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{array}\right] \xrightarrow{-1R_1+R_2} C_2 = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 3 & 1 & 5 & -7 & 1 \end{array}\right]$$

$$\xrightarrow{-3R_1+R_3} C_3 = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 0 & -5 & 5 & -10 & -20 \end{array}\right]$$

$$\xrightarrow{-1R_2} C_4 = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & -5 & 5 & -10 & -20 \end{array}\right]$$

$$\xrightarrow{5R_2+R_3} C_5 = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_6+R_1} C_6 = \left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We see that (C_6) is the augmented matrix representation of system

$$(S_6): \left\{\begin{array}{rcrcrcrcr} 1x_1 & + & 0x_2 & + & 2x_3 & + & (-3)x_4 & = & -1 \\ 0x_1 & + & 1x_2 & + & (-1)x_3 & + & 2x_4 & = & 4 \\ 0x_1 & + & 0x_2 & + & 0x_3 & + & 0x_4 & = & 0 \end{array}\right.$$

from which we can read off the solutions of (S).

This is why, and how, augmented matrix representations and row operations can be used as a short-hand for presenting the manipulations for solving systems of linear equations.

(d) Overall, when asked to find all solutions of the given system (S), we may present the manipulations as follows:—

Consider the system

$$(S): \left\{\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & & & + & x_4 & = & 7 \\ x_1 & + & x_2 & + & x_3 & - & x_4 & = & 3 \\ 3x_1 & + & x_2 & + & 5x_3 & - & 7x_4 & = & 1 \end{array}\right.$$

Denote by C_1 its augmented matrix representation. Apply the sequence of operations on C_1:

$$C_1 = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{array}\right] \xrightarrow{-1R_1+R_2} C_2 = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 3 & 1 & 5 & -7 & 1 \end{array}\right]$$

$$\xrightarrow{-3R_1+R_3} C_3 = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 0 & -5 & 5 & -10 & -20 \end{array}\right] \xrightarrow{-1R_2} C_4 = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & -5 & 5 & -10 & -20 \end{array}\right]$$

$$\xrightarrow{5R_2+R_3} C_5 = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_6+R_1} C_6 = \left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Note that C_6 is the augmented matrix representation of system

$$(S'): \left\{\begin{array}{rcrcrcrcr} x_1 & & & + & 2x_3 & - & 3x_4 & = & -1 \\ & & x_2 & - & x_3 & + & 2x_4 & = & 4 \\ & & & & & & 0 & = & 0 \end{array}\right.$$

(From (S') we read off the relations $x_1 = -1 - 2x_3 + 3x_4$ and $x_2 = 4 + x_3 - 2x_4$.)

Hence the solutions of (S) are described by $(x_1, x_2, x_3, x_4) = (-1 - 2s + 3t, 4 + s - 2t, s, t)$ where s, t are arbitrary numbers.

(e) There is something special about the matrix C_6 in this manipulations. In the context here, it somehow signifies that we are ready to 'read off' all the solutions of (S). We will look at it in greater detail later.

(f) What we are doing in this illustration applies in the general situation as well.

3. Recall the definition for the notion of *equivalent systems*:

Let $(S), (T)$ be two systems of m linear equations with n unknowns.

We say (S) is equivalent to (T) as systems if and only if every solution of (S) is a solution of (T) and every solution of (T) is a solution of (S).

4. **Theorem (1). (Sufficiency Condition for equivalence of systems of linear equations.)**

   Let $(S), (T)$ be two systems of m linear equations with n unknowns.

   Let C, D be the respective augmented matrix representations of $(S), (T)$.

   Suppose C is row-equivalent to D. Then (S) is equivalent to (T).

   Proof. The argument relies on Lemma $(\star)$. Refer to Appendix.

   Remark. Theorem (1) justifies, at the theoretical level, the use of augmented matrix representations and row operations in presenting the manipulations towards determining all solutions of a given system of linear equations.

5. **Illustrations.**

   We now re-worked some examples we met earlier.

   (a) Refer to Example (2) of the handout *What is solving a system of linear equations.*

   Consider the system

$$(S): \left\{ \begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & = & 4 \\ x_1 & + & 3x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & 6 \end{array} \right.$$

   Denote by C_1 its augmented matrix representation. Apply the sequence of operations on C_1:

$$C_1 = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{array} \right] \xrightarrow{-1R_1+R_2} C_2 = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 2 & 6 & 5 & 6 \end{array} \right]$$

$$\xrightarrow{-2R_1+R_3} \quad C_3 = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 \end{array} \right] \xrightarrow{-2R_2+R_3} C_4 = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -4 \end{array} \right]$$

$$\xrightarrow{-1R_3} \quad C_5 = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{array} \right] \xrightarrow{-2R_2+R_1} C_6 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{array} \right] \xrightarrow{-1R_3+R_2} C_7 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array} \right]$$

   C_7 is the augmented matrix representation of the system

$$(S'): \left\{ \begin{array}{ccccr} x_1 & & & = & 2 \\ & x_2 & & = & -3 \\ & & x_3 & = & 4 \end{array} \right.$$

   (C_1 and C_7 are row-equivalent. Then (S) and (S') are equivalent systems.
   From (S') we read off the relations $x_1 = 2$, $x_2 = -3$, $x_3 = 4$.)
   Hence the (only) solution of (S) is given by $(x_1, x_2, x_3) = (2, -3, 4)$.

   (b) Refer to Example (3) of the handout *What is solving a system of linear equations.*

   Consider the system

$$(S): \left\{ \begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 2 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 7 \\ -x_1 & + & 3x_2 & - & 5x_3 & = & 3 \end{array} \right.$$

   Denote by C_1 its augmented matrix representation. Apply the sequence of operations on C_1:

$$C_1 = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 3 & -2 & 1 & 7 \\ -1 & 3 & -5 & 3 \end{array} \right] \xrightarrow{-3R_1+R_2} C_2 = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ -1 & 3 & -5 & 3 \end{array} \right]$$

$$\xrightarrow{1R_1+R_3} \quad C_3 = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & -4 & 5 \end{array} \right] \xrightarrow{-2R_2+R_3} C_4 = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

   C_4 is the augmented matrix representation of the system

$$(S'): \left\{ \begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 2 \\ & & x_2 & - & 2x_3 & = & 1 \\ & & & & 0 & = & 1 \end{array} \right.$$

   (C_1 and C_4 are row-equivalent. Then (S) and (S') are equivalent systems.
   (S') has no solution.)
   Hence (S) has no solution.

(c) Refer to Example (4) of the handout *What is solving a system of linear equations.*

Consider the system

$$(S): \left\{ \begin{array}{rcrcrcr} & & x_2 & - & 2x_3 & = & 1 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & -4 \\ 2x_1 & + & 7x_2 & - & 12x_3 & = & 11 \end{array} \right.$$

Denote by C_1 its augmented matrix representation. Apply the sequence of operations on C_1:

$$C_1 = \left[\begin{array}{ccc|c} 0 & 1 & -2 & 1 \\ -1 & -2 & 3 & -4 \\ 2 & 7 & -12 & 11 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} C_2 = \left[\begin{array}{ccc|c} -1 & -2 & 3 & -4 \\ 0 & 1 & -2 & 1 \\ 2 & 7 & -12 & 11 \end{array} \right]$$

$$\xrightarrow{-1R_1} \quad C_3 = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 1 \\ 2 & 7 & -12 & 11 \end{array} \right] \xrightarrow{-2R_1+R_3} C_4 = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 3 & -6 & 3 \end{array} \right]$$

$$\xrightarrow{-3R_2+R_3} \quad C_5 = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{-2R_2+R_1} C_6 = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

C_6 is the augmented matrix representation of the system

$$(S'): \left\{ \begin{array}{rcrcrcr} x_1 & & & + & x_3 & = & 2 \\ & & x_2 & - & 2x_3 & = & 1 \\ & & & & 0 & = & 0 \end{array} \right.$$

(C_1 and C_6 are row-equivalent. Then (S) and (S') are equivalent systems.
From (S') we read off the relations $x_1 = 2 - x_3$ and $x_2 = 1 + 2x_3$.)
Hence the solutions of (S) is described by $(x_1, x_2, x_3) = (2 - t, 1 + 2t, t)$, where t is an arbitrary real number.

6. **Another application of Theorem (1).**

Below is a re-formulation of Theorem (1). It provides a useful method for verifying two given matrices of the same size are not row-equivalent.

Corollary (2).

Let $(S), (T)$ be two systems of m linear equations with n unknowns.

Let C, D be the respective augmented matrix representations of $(S), (T)$.

Suppose (S) is not equivalent to (T). Then C is not row-equivalent to D.

Remark. Such a re-formulation of Theorem (1) is known as the 'contra-positive' re-formulation of the statement of Theorem (1).

7. **Examples. (Applications of Corollary (2).)**

(a) *Let* $C = \left[\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{array} \right]$, $D = \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right]$.

We verify that C is not row-equivalent to D:
Note that C is the augmented matrix representation of

$$(S): \left\{ \begin{array}{rcrcrcr} x_1 & & & + & x_3 & = & 1 \\ & & x_2 & & & = & 1 \end{array} \right.$$

and D is the augmented matrix representation of

$$(T): \left\{ \begin{array}{rcrcrcr} x_1 & & & + & x_3 & = & 0 \\ & & x_2 & & & = & 0 \end{array} \right.$$

$(x_1, x_2, x_3) = (0, 0, 0)$ is a solution of (T).
$(x_1, x_2, x_3) = (0, 0, 0)$ is not a solution of (S).
Hence $(S), (T)$ are not equivalent systems. It follows that C is not row-equivalent to D.

(b) *Let* $C = \left[\begin{array}{cccc} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{array} \right]$, $D = \left[\begin{array}{cccc} 1 & 2 & 7 & 4 \\ 0 & 1 & 2 & 1 \\ 2 & 1 & 8 & 5 \end{array} \right]$.

We verify that C is not row-equivalent to D:

Note that C is the augmented matrix representation of

$$(S): \left\{\begin{array}{ccccccc} x_1 & + & x_2 & + & 2x_3 & = & 3 \\ & & x_2 & + & x_3 & = & 2 \\ x_1 & & & + & x_3 & = & 1 \end{array}\right.$$

and D is the augmented matrix representation of

$$(T): \left\{\begin{array}{ccccccc} x_1 & + & 2x_2 & + & 7x_3 & = & 4 \\ & & x_2 & + & 2x_3 & = & 1 \\ 2x_1 & & & + & 8x_3 & = & 5 \end{array}\right.$$

The solutions of (S) are described by $(x_1, x_2, x_3) = (1 - t, 2 - t, t)$ where t is an arbitrary number.
In particular $(x_1, x_2, x_3) = (1, 2, 0)$ is a solution of (S).
Note that $(x_1, x_2, x_3) = (1, 2, 0)$ is not a solution of (T). (Why?)
Hence $(S), (T)$ are not equivalent systems. It follows that C is not row-equivalent to D.

8. **Warning.**

Theorem (1) is 'one-way' in the sense that

there are systems which are equivalent systems whose respective augmented matrix representations are not row-equivalent.

In fact, when the respective augmented matrix representations of two systems are not row-equivalent, we cannot draw any conclusion for the two systems, on the question whether they are consistent or not.

Examples.

(a) Let $(S), (T)$ be the systems given by

$$(S) \quad \left\{\begin{array}{ccccc} x_1 & + & x_2 & = & 0 \\ & & 0 & = & 1 \end{array}\right. \quad , \qquad (T) \left\{\begin{array}{ccccc} x_1 & - & x_2 & = & 0 \\ & & 0 & = & 1 \end{array}\right.$$

Both $(S), (T)$ are inconsistent. So they are equivalent.
The respective augmented matrices of $(S), (T)$ are

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It can be shown that C, D are not row-equivalent.

(b) Let $(S), (T)$ be the systems given by

$$(S) \quad \left\{\begin{array}{ccccc} x_1 & + & x_2 & = & 0 \\ & & 0 & = & 0 \end{array}\right. \quad , \qquad (T) \left\{\begin{array}{ccccc} x_1 & - & x_2 & = & 0 \\ & & 0 & = & 1 \end{array}\right.$$

(S) is consistent but (T) is inconsistent. So they are not equivalent.
The respective augmented matrices of $(S), (T)$ are

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It can be shown that C, D are not row-equivalent.

Remark. At this moment we have not made the appropriate preparation for an simple justification for the claims on the non-row-equivalence for the respective matrices in these examples. It will transpire that were C, D two row-equivalent matrices with two rows, there would be a (2×2)-square matrix H which satisfied $C = HD$. But then such a matrix H would necessarily be the identity matrix. This would lead to a contradiction.

Theorem (3).

Let $(S), (T)$ be two systems of m linear equations with n unknowns.

Let C, D be the respective augmented matrix representations of $(S), (T)$.

Suppose $(S), (T)$ are consistent. Then the following statements are logically equivalent:

(†) *(S) is equivalent to (T).*

(‡) *C is row-equivalent to D.*

Proof. Omitted. (One possible argument relies on the notions of *inner product* and *orthogonal complement*, which will touch upon towards the end of this course.)

9. **Appendix: How to prove Theorem (1).**

Recall, from the handout *Basic terminologies on systems of linear equations*, the result below, which we now denote as **Lemma ($\star$)**:

> *Let $(S), (T)$ be two systems of m linear equations with n unknowns. The statements below hold:*
>
> (a) *Suppose (T) is obtained by (S) by the application of one equation operation. Then (S) is equivalent to (T).*
>
> (b) *Suppose (T) is obtained by (S) by the application of a finite sequence of equation operations. Then (S) is equivalent to (T).*

With the help of Lemma ($\star$) we prove Theorem (1):

By assumption, there is a finite sequence of $(m \times (n+1))$-matrices starting from C and ending at D, say,

$$C = C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow \cdots \longrightarrow C_{N-1} \longrightarrow C_N = D$$

with each step '$C_k \longrightarrow C_{k+1}$' being given by a row operation, labeled ρ_k.

Read each C_k as a system of m linear equations with n unknowns, which we label as (S_k).

Then, for each k, (S_{k+1}) is obtained from (S_k) by an application of the equation operation ε_k which corresponds to the row operation ρ_k.

Now we have obtained the sequence of systems of equations

$$(S) \longrightarrow (S_2) \longrightarrow (S_3) \longrightarrow \cdots \longrightarrow (S_{N-1}) \longrightarrow (T)$$

with each step '$(S_k) \longrightarrow (S_{k+1})$' being given by the equation operation ε_k.

By Lemma ($\star$), we conclude that $(S), (T)$ are equivalent.