MATH1030 Basic terminologies on systems of linear equations

1. **Definition. (Systems of linear equations.)**

   Let a_{ij} be (fixed) real numbers for each $i = 1, \cdots, m$ and for each $j = 1, \cdots, n$.

   Let b_k be (fixed) real numbers for each $k = 1, \cdots, m$.

   (a) *The system of m simultaneous equations with unknowns $x_1, x_2, \cdots, x_n$*

   $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

   is called a system of m linear equations with n unknowns.

   The numbers a_{ij}'s, b_k's are referred to as givens in this system of linear equations.

   (b) *Denote such a system of linear equations by (S).*

   Let $t_1, t_2, \cdots, t_n$ be (fixed) real numbers.

   We say $(x_1, x_2, \cdots, x_n) = (t_1, t_2, \cdots, t_n)$ is a solution of the system (S) if and only if the m equalities

   $$\begin{array}{c} a_{11}t_1 + a_{12}t_2 + \cdots + a_{1n}t_n = b_1, \\ a_{21}t_1 + a_{22}t_2 + \cdots + a_{2n}t_n = b_2, \\ \vdots \\ a_{m1}t_1 + a_{m2}t_2 + \cdots + a_{mn}t_n = b_m \end{array}$$

   hold simultaneously.

   (c) *(Again denote such a system of linear equations by (S).)*

      i. *We say (S) is consistent if and only if there is some solution for (S).*

      ii. *We say (S) is inconsistent if and only if there is no solution for (S).*

2. **Definition. (Equation operation 'adding a scalar multiple of one equation to another'.)**

   Consider the system of linear equations

   $$(S) : \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

   in which $x_1, x_2, \cdots, x_n$ are the unknowns.

   Suppose α is a real number.

   When we replace the k-th equation

   $$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = b_k$$

   of (S) by the equation

   $$(\alpha a_{i1} + a_{k1})x_1 + (\alpha a_{i2} + a_{k2})x_2 + \cdots + (\alpha a_{in} + a_{kn})x_n = \alpha b_i + b_k,$$

   in which $i \neq k$, to obtain some (other) system, we say we are applying the equation operation '$\alpha \times (i) + (k)$' to (S).

   Such an equation operation is called 'adding a scalar multiple of one equation of (S) to another equation of (S)'.

3. **Definition. (Equation operation 'multiplying a non-zero scalar to one equation'.)**

   Consider the system of linear equations

   $$(S) : \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

   in which $x_1, x_2, \cdots, x_n$ are the unknowns.

   Suppose β is a non-zero real number.

When we replace the k-th equation

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = b_k$$

of (S) by the equation

$$\beta a_{k1}x_1 + \beta a_{k2}x_2 + \cdots + \beta a_{kn}x_n = \beta b_k,$$

to obtain some (other) system, we say we are applying the equation operation '$\beta \times \textcircled{k}$' to (S).

Such an equation operation is called 'multiplying a non-zero scalar to one equation of (S)'.

4. **Definition. (Equation operation 'interchanging two equations'.)**

   Consider the system of linear equations

$$(S): \left\{ \begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \right.$$

   in which $x_1, x_2, \cdots, x_n$ are the unknowns.

   When we interchange the i-th equation

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

   of (S) and the k-th equation

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = b_k$$

   of (S), in which $i \neq k$, to obtain some (other) system, we say we are applying the equation operation '$\textcircled{i} \leftrightarrow \textcircled{k}$' to (S).

   Such an equation operation is called 'interchanging two equations of (S)'.

5. **Definition. (Equation operations.)**

   Let $(S), (T)$ be systems of m linear equations with n unknowns.

   We say we are applying one equation operation on (S) to obtain the system (T) if and only if (T) is the resultant of the application of

   - *one equation operation 'adding a scalar multiple of one equation of (S) to another', or*
   - *one equation operation 'multiplying a non-zero scalar to one equation (S)', or*
   - *one equation operation 'interchanging two equations of (S)'.*

6. **Definition. (Equivalent systems of linear equations.)**

   Let $(S), (T)$ be systems of m linear equations with n unknowns.

   We say (S) is equivalent to (T) as systems if and only if both statements below hold:

   (a) *Every solution of (S) is a solution of (T).*

   (b) *Every solution of (T) is a solution of (S).*

7. **Theorem (1).**

   Let $(S), (T)$ be systems of m linear equations with n unknowns.

   (a) *Suppose (T) is resultant from the application of one equation operation on (S). Then (S) is equivalent to (T) as systems.*

   (b) *Suppose (T) is resultant from the application of finitely many equation operations, starting from (S). Then (S) is equivalent to (T) as systems.*

   Proof. A tedious (but easy) word game on the definitions.

8. **Theorem (2).**

   The statements below hold:

   (a) *Suppose (S) is a system of m linear equations with n unknowns. Then (S) is equivalent to (S) as systems.*

   (b) *Let $(S), (T)$ be systems of m linear equations with n unknowns.*
   Suppose (S) is equivalent to (T) as systems. Then (T) is equivalent to (S) as systems.

   (c) *Let $(S), (T), (U)$ be systems of m linear equations with n unknowns.*
   Suppose (S) is equivalent to (T) as systems, and (T) is equivalent to (U) as systems. Then (S) is equivalent to (U) as systems.

   Proof. A tedious (but easy) word game on the definitions.

9. **Illustration of the relevance of Theorem (1) through a concrete example.**

Consider the question:

What are the solutions of the system of linear equations

$$(S)\quad \left\{\begin{array}{rcrcrcr} & & x_2 & - & 2x_3 & = & 1 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & -4 \\ 2x_1 & + & 7x_2 & - & 12x_3 & = & 11 \end{array}\right.$$

with unknowns x_1, x_2, x_3 in the reals, if there is any at all?

(a) We can answer this question through the three-step process:—

- *Step 1. Searching for 'candidate solution' for the system of equations.*
  Suppose (x_1, x_2, x_3) is a solution of (S). Then blah-blah-blah. Therefore it is possible for (x_1, x_2, x_3) to be $(2-t, 1+2t, t)$ for some real number t, and there is no other possibility.
- *Step 2. Checking 'candidate solution'.*
  Suppose $(x_1, x_2, x_3) = (2-t, 1+2t, t)$ for some real number t. Then blah-blah-blah. Therefore (x_1, x_2, x_3) is indeed a solution of the system concerned.
- *Step 3. Drawing conclusion.*
  The solutions of the system concerned is given by $(x_1, x_2, x_3) = (2-t, 1+2t, t)$ where t are arbitrary numbers.

(b) The manipulation in the 'blah-blah-blah' in *Step 1* is this chain of manipulation on the symbols x_1, x_2, x_3, which stand for some concrete real numbers in *Step 1*:

$$(S_1)\quad \left\{\begin{array}{rcrcrcrl} & & x_2 & - & 2x_3 & = & 1 & \text{—— ①} \\ -x_1 & - & 2x_2 & + & 3x_3 & = & -4 & \text{—— ②} \\ 2x_1 & + & 7x_2 & - & 12x_3 & = & 11 & \text{—— ③} \end{array}\right.$$

$$\text{①} \leftrightarrow \text{②}:\quad (S_2)\quad \left\{\begin{array}{rcrcrcrl} -x_1 & - & 2x_2 & + & 3x_3 & = & -4 & \text{—— ④} \\ & & x_2 & - & 2x_3 & = & 1 & \text{—— ⑤} \\ 2x_1 & + & 7x_2 & - & 12x_3 & = & 11 & \text{—— ③} \end{array}\right.$$

$$(-1) \times \text{④}:\quad (S_3)\quad \left\{\begin{array}{rcrcrcrl} x_1 & + & 2x_2 & - & 3x_3 & = & 4 & \text{—— ⑥} \\ & & x_2 & - & 2x_3 & = & 1 & \text{—— ⑤} \\ 2x_1 & + & 7x_2 & - & 12x_3 & = & 11 & \text{—— ③} \end{array}\right.$$

$$(-2) \times \text{⑥} + \text{③}:\quad (S_4)\quad \left\{\begin{array}{rcrcrcrl} x_1 & + & 2x_2 & - & 3x_3 & = & 4 & \text{—— ⑥} \\ & & x_2 & - & 2x_3 & = & 1 & \text{—— ⑤} \\ & & 3x_2 & - & 6x_3 & = & 3 & \text{—— ⑦} \end{array}\right.$$

$$(-3) \times \text{⑤} + \text{⑦}:\quad (S_5)\quad \left\{\begin{array}{rcrcrcrl} x_1 & + & 2x_2 & - & 3x_3 & = & 4 & \text{—— ⑥} \\ & & x_2 & - & 2x_3 & = & 1 & \text{—— ⑤} \\ & & & & 0 & = & 0 & \text{—— ⑧} \end{array}\right.$$

$$(-2) \times \text{⑤} + \text{⑥}:\quad (S_6)\quad \left\{\begin{array}{rcrcrcrl} x_1 & & & + & x_3 & = & 2 & \text{—— ⑨} \\ & & x_2 & - & 2x_3 & = & 1 & \text{—— ⑤} \\ & & & & 0 & = & 0 & \text{—— ⑧} \end{array}\right.$$

(c) Now we re-interpret this chain of manipulation that we write in *Step 1* (in search of 'candidate solutions' of (S)).

Regard the symbols x_1, x_2, x_3 as unknowns arising from the system (S) throughout the manipulation.

Then $(S_1), (S_2), \cdots, (S_6)$ are just the successive systems resultant from an application of equations operations, starting with the system (S).

Theorem (1) tells us that each of $(S_1), (S_2), \cdots, (S_6)$ is equivalent to each other. So the solutions of (S_1) and (S_6) are the same.

We can read off the solutions of (S_6) easily (from the relations $x_1 = 2 - x_3$, $x_2 = 1 + 2x_3$), and hence of (S) itself.

So having presented the manipulation done in *Step 1* above (and interpreting the manipulation as applications of equation operations), we may jump directly to the conclusion stated in *Step 3*:

The solutions of the system concerned is given by $(x_1, x_2, x_3) = (2-t, 1+2t, t)$ where t are arbitrary numbers.