THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 6 Solutions

Problems that may be demonstrated in class :

Q1. By using mean value theorem, show that

$$|\cos x - \cos y| \leq |x - y|$$

for all $x,\ y \in \mathbb{R}$

Q2. Let $a,\ b \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(x) > 0$ for all $x \in [a, b]$. Show that f is increasing on (a, b) by using mean value theorem.

Q3. Consider the equation $\cos x = 2x$.
(a) Show that the equation has at least 1 solution.
(b) Show that the equation has at most 1 solution.

Q4. Let $f : [a, b] \to \mathbb{R}\backslash\mathbb{Q}$ be continuous. Prove that f must be a constant function.

Q5. Let $f : [0, 1] \to (0, 1)$ be a continuous function. Show that f has a fixed point in $(0, 1)$. i.e.

$$\exists\, c \in (0, 1) \text{ such that } f(c) = c$$

Q6. (a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or minimum on $\mathbb{R}$.
(b) Give an example such that f attains either maximum or minimum, but not both.

Q7. Find the Taylor polynomial of degree 4 of the following functions at $x = 0$
(a) $\ln(1 + x)$
(b) $(1 + x) \ln(1 + x)$

Q8. (a) Find the Taylor series of $f(x) = \dfrac{1}{1 - x}$
(b) What is the radius of convergence, R?
(c) Is the Taylor series absolutely convergent when $x = R$ and $x = -R$ respectively?

Solution

Q1. Let $x, y \in \mathbb{R}$, and $f(z) = \cos z\ \forall z$. If $x = y$, the statement clearly holds.

For $x \neq y$, by mean value theorem, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(x) - f(y)}{x - y}$$

Therefore we have

$$|\sin c| = \left|\frac{\cos x - \cos y}{x - y}\right|$$

and thus

$$\left|\frac{\cos x - \cos y}{x - y}\right| \leq 1 \quad \text{and} \quad |\cos x - \cos y| \leq |x - y|$$

using the fact that $|\sin x| \leq 1\ \forall x \in \mathbb{R}$

Q2. Let $x, y \in (a, b)$ with $x < y$. By mean value theorem, there exists $c \in (x, y) \subset (a, b)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c) > 0$$

We then obtain

$$f(y) - f(x) > 0$$

since $y - x > 0$. Therefore f is increasing on (a, b)

Q3. (a) Let $f(x) = \cos(x) - 2x$. Note that $f(\frac{\pi}{2}) = -\pi < 0$ and $f(-\frac{\pi}{2}) = \pi > 0$. By intermediate value theorem, there exists $c \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that

$$f(c) = 0$$

(b) Suppose there exists c_1, c_2 with $c_1 < c_2$ such that $f(c_1) = f(c_2) = 0$. By mean value theorem, there exists $d \in (c_1, c_2)$ such that $f'(d) = \dfrac{f(c_1) - f(c_2)}{c_1 - c_2} = 0$ i.e.

$$-\sin d = 2$$

which is impossible. Therefore the above equation can have one and only one solution.

Q4. Suppose f is not a constant function. Then there exists $c_1, c_2 \in [a, b]$, with $c_1 \neq c_2$, such that $f(c_1) < f(c_2)$. Note that for any two irrational numbers x, y with $x < y$, we can always find $c \in \mathbb{Q}$ such that $x < c < y$. Let $c \in \mathbb{Q}$ such that $c \in (f(c_1), f(c_2))$. By intermetidate value theorem, there exists $\xi \in (c_1, c_2)$ such that $f(\xi) = c$. It is impossible since $c \notin \mathbb{R}\backslash\mathbb{Q}$

Q5. Let $g(x) = f(x) - x$ for all $x \in [0, 1]$. We have $g(1) = f(1) - 1 < 0$ and $g(0) = f(0) > 0$. By intermediate value theorem, there exists $c \in (0, 1)$ such that $g(c) = 0$. i.e. $f(c) = c$.

Q6. Assume f is not identically zero (otherwise maximum = minimum = 0). Choose $c \in \mathbb{R}$ such that $f(c) \neq 0$. Since $\lim_{x \to \pm\infty} f(x) = 0$, there exists $N \in \mathbb{R}$ such that $|f(x)| \leq \dfrac{|f(c)|}{2}$ for all $|x| > N$. Note that $c \in [-N, N]$ by construction. Consider the interval $[-N, N]$. By extreme value theorem,

$$\exists\, \alpha, \beta \text{ such that } f(\alpha) \leq f(x) \leq f(\beta) \quad \forall x \in [-N, N]$$

Since f is bounded on $[-N, N]$ and also bounded when $|x| > N$. f is bounded on $\mathbb{R}$.
(i) Supoose $f(c) > 0$. For all $|x| > N$ we have

$$f(x) \leq \frac{f(c)}{2} \leq f(c) \leq f(\beta) \quad (\because c \in [-N, N])$$

Hence $f(x) \leq f(\beta)$ for all $x \in \mathbb{R}$.
(ii) Supoose $f(c) < 0$. For all $|x| > N$ we have

$$f(x) \geq -\frac{|f(c)|}{2} \geq -|f(c)| = f(c) \geq f(\alpha) \quad (\because c \in [-N, N])$$

Hence $f(x) \geq f(\alpha)$ for all $x \in \mathbb{R}$.

Q7. (a) Since $f(0) = 0, f'(0) = 1, f''(0) = -1, f^{(3)}(0) = 2, f^{(4)}(0) = -6$, we then have

$$T_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

(b) Let $S_4(x)$ be the Taylor polynomial of $(1+x)\ln(1+x)$. By (a), let $S(x) = (1+x)\cdot T_4(x)$

$$S(x) = (1+x)(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}) = x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{4}$$

Therefore $S_4(x) = x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12}$

Q8. (a) Note that $f^{(n)}(0) = n!$. Therefore

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} x^n = \sum_{n=0}^{\infty} x^n$$

(b) Note that the partial sum is

$$S_n(x) = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$$

First it is easy to see that S_n diverges if $|x| > 1$. Next, consider the following series:

$$\sum_{k=0}^{n} |x^k|$$

For $0 \le x < 1$,

$$\sum_{k=0}^{n} |x^k| = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x} \to \frac{1}{1 - x} \quad \text{as } n \to \infty$$

For $-1 < x < 0$, we have $0 < y < 1$, where $y = -x$. Hence

$$\sum_{k=0}^{n} |x^k| = \sum_{k=0}^{n} |(-1)^k y^k| = \frac{1 - y^{n+1}}{1 - y} \to \frac{1}{1 + x} \quad \text{as } n \to \infty$$

Therefore we can conclude that $S(x)$ is absolutely convergent if $|x| < 1$ and thus $R = 1$.
(c) If $x = 1$, $S_n = \sum_{k=0}^{n} 1$, which does not converge. If $x = -1$, $S_n = \sum_{k=0}^{n} (-1)^k$, which is also divergent.