THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 6

Theorem (Intermeidate value theorem)

Suppose $f(x)$ is a continuous function on a closed and bounded interval $[a, b]$. Then for any real number c between $f(a)$ and $f(b)$ (exclusive), there exists $\zeta \in (a, b)$ such that $f(\zeta) = c$.

Theorem (Extreme value theorem)

Suppose $f(x)$ is a continuous function on a closed and bounded interval $[a, b]$. Then there exists $\alpha, \beta \in [a, b]$ such that for any $x \in [a, b]$, we have

$$f(\alpha) \leq f(x) \leq f(\beta)$$

Theorem (Mean value theorem)

Suppose a, b are real numbers and $a < b$.

1. *Lagrange's:* Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

2. *Cauchy's:* Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0 \, \forall x \in (a, b)$. Then there exists $\xi \in (a, b)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Definition (Taylor Polynomial)

Let $f(x)$ be a function such that the n-th derivative exists at $x = a$. The *Taylor polynomial* of degree n of $f(x)$ at $x = a$ is the polynomial

$$p_n(x) = f(a) + f'(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k$$

Definition (Taylor series)

Let $f(x)$ be an infinitely differentiable function. The *Taylor series* of $f(x)$ at $x = a$ is the infinite power series

$$T(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f^{(3)}(a)}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k$$

Theorem (Radius of convergence) For any power series

$$S(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

There exists $R \in [0, +\infty]$ called radius of convergence such that

1. $S(x)$ is absolutely convergent for any $|x| < R$ and divergent for any $|x| > R$. For $|x| = R$, $S(x)$ may or may not be convergent.
2. When $S(x)$ is considered as a function of x, it is differentiable on $(-R, R)$ and its derivative is

$$S'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + \cdots$$

Problems that may be demonstrated in class :

Q1. By using mean value theorem, show that

$$|\cos x - \cos y| \leq |x - y|$$

for all $x,\ y \in \mathbb{R}$

Q2. Let $a,\ b \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(x) > 0$ for all $x \in [a, b]$. Show that f is increasing on (a, b) by using mean value theorem.

Q3. Consider the equation $\cos x = 2x$.
(a) Show that the equation has at least 1 solution.
(b) Show that the equation has at most 1 solution.

Q4. Let $f : [a, b] \to \mathbb{R}\backslash\mathbb{Q}$ be continuous. Prove that f must be a constant function.

Q5. Let $f : [0, 1] \to (0, 1)$ be a continuous function. Show that f has a fixed point in $(0, 1)$. i.e.

$$\exists\, c \in (0, 1) \text{ such that } f(c) = c$$

Q6. (a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or minimum on $\mathbb{R}$.
(b) Give an example such that f attains either maximum or minimum, but not both.

Q7. Find the Taylor polynomial of degree 4 of the following functions at $x = 0$
(a) $\ln(1 + x)$
(b) $(1 + x)\ln(1 + x)$

Q8. (a) Find the Taylor series of $f(x) = \dfrac{1}{1 - x}$
(b) What is the radius of convergence, R?
(c) Is the Taylor series absolutely convergent when $x = R$ and $x = -R$ respectively?