**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH1010D&E (2016/17 Term 1)**
**University Mathematics**
**Tutorial 4 Solutions**

**Problems that may be demonstrated in class :**

Q1. Find the limit of the following expressions.

(a) $\lim_{x\to 0} \dfrac{\ln(x^4+x^2+3)}{\ln(x^6+6x^4+9)}$ (b) $\lim_{x\to \infty} \dfrac{\ln(x^4+x^2+3)}{\ln(x^6+6x^4+9)}$

(c) $\lim_{x\to\infty} e^{-x}\tanh x$ (d) $\lim_{x\to 0} \dfrac{(\ln(\sec(x)))^2}{\cos(\sin(\tan(x)))}$

(e) $\lim_{x\to 0} \dfrac{\sin(e^{2x^2}-1)}{e^{2x^2}-1}$ (f) $\lim_{x\to 0} x\cos\left(\dfrac{1}{x}\right)$

Q2. Determine where the function $f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$ is continuous.

Q3. Given that the function $f(x) = \begin{cases} e^{-\frac{1}{x}}\sin\left(\cos\frac{1}{x}\right) & \text{if } x>0 \\ a & \text{if } x=0 \\ be^{e^{\sin\frac{1}{x}}} & \text{if } x<0 \end{cases}$ is continuous on $\mathbb{R}$ where $a, b\in\mathbb{R}$. Find the value of $a$ and $b$.

Q4. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(x) = f(x/2)$ for all real number $x$. Show that $f(x) = f(0)$ for all real number $x$.

Q5. Show that every (univariate) polynomial of odd degree (with real coefficients) has a root in real numbers.

Q6. Consider $f(x) = \begin{cases} \sqrt[2016]{x}\sin\left(2e^{1/x}\right)+1 & \text{if } x>0 \\ 1 & \text{if } x\leq 0 \end{cases}$.

(a) Show that $f$ is a continuous function on $\mathbb{R}$.

(b) Prove that $f$ has a fixed point in the interval $[0,1]$. (That is, there exists $x\in[0,1]$ such that $f(x)=x$)

**Solutions:** Q1. (a) Notice that $\lim_{x\to 0} x^4+x^2+3 = 3 > 0$ and $\lim_{x\to 0} x^6+6x^4+9 = 9 > 0$. So we can use the continuity of ln to get $\lim_{x\to 0}\ln(x^4+x^2+3) = \ln 3$ and $\lim_{x\to 0}\ln(x^6+6x^4+9) = \ln 9 \neq 0$. Therefore we have

$$\lim_{x\to 0} \frac{\ln(x^4+x^2+3)}{\ln(x^6+6x^4+9)} = \frac{\ln 3}{\ln 9} = \frac{1}{2}.$$

(b) Since we are looking for limit when $x$ tends to infinity, we may just consider $x>1$ and sufficiently large. So

$$\ln(x^4+x^2+3) = \ln(x^4) + \ln(1+x^{-2}+3x^{-4}) = 4\ln x + \ln(1+x^{-2}+3x^{-4})$$

and

$$\ln(x^6+6x^4+9) = \ln(x^6) + \ln(1+6x^{-2}+9x^{-6}) = 6\ln x + \ln(1+6x^{-2}+9x^{-6}).$$

Now we have

$$\frac{\ln(x^4+x^2+3)}{\ln(x^6+6x^4+9)} = \frac{4\ln x + \ln(1+x^{-2}+3x^{-4})}{6\ln x + \ln(1+6x^{-2}+9x^{-6})} = \frac{4+\frac{\ln(1+x^{-2}+3x^{-4})}{\ln x}}{6+\frac{\ln(1+6x^{-2}+9x^{-6})}{\ln x}}$$

Now use the fact that $\lim_{x\to\infty} x^{-1} = 0$, $\lim_{x\to\infty} \dfrac{1}{\ln x} = 0$ and ln is continuous, we have $\lim_{x\to\infty} \frac{\ln(1+x^{-2}+3x^{-4})}{\ln x} = \ln 1 \cdot 0 = 0$ and $\lim_{x\to\infty} \frac{\ln(1+6x^{-2}+9x^{-6})}{\ln x} = \ln 1 \cdot 0 = 0$. Therefore we have

$$\lim_{x\to\infty} \frac{\ln(x^4+x^2+3)}{\ln(x^6+6x^4+9)} = \frac{4}{6} = \frac{2}{3}.$$

(c) Knowing that $|\tanh x| \leq 1$ for all real number $x$ and $\lim_{x\to\infty} e^{-x} = 0$, the limit is 0.

(d) This is a simple application of continuity. All functions that appear are continuous in some range. We have $\lim_{x\to 0}\sec(x) = 1$, $\lim_{x\to 1}\ln x = 0$, $\lim_{x\to 0}\tan x = 0$, $\lim_{x\to 0}\sin x = 0$, and $\lim_{x\to 0}\cos x = 1$. So the limit is 0.

(e) Using the fact that $e^{2x^2}-1$ is continuous at $x=0$ and $\lim_{x\to 0}\dfrac{\sin x}{x} = 1$, we have $\lim_{x\to 0} e^{2x^2}-1 = e^0 - 1 = 0$ and $\lim_{x\to 0}\dfrac{\sin(e^{2x^2}-1)}{e^{2x^2}-1} = 1$.

(f) Since for $x \neq 0$ $|\cos(\frac{1}{x})| \leq 1$ and $\lim_{x\to 0} x = 0$, we have

$$\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0.$$

Q2. For $a \neq 0$, $x \mapsto \dfrac{1}{x}$, $x \mapsto \sin x$ and $x \mapsto x$ are continuous at $x = a$. Therefore the function $f(x)$ is continuous on $\mathbb{R}/\{0\}$. At $x=0$, we have $\lim_{x\to 0} x\sin\frac{1}{x} = 0 = f(0)$. Therefore $f$ is continuous at 0 too. So $f$ is continuous on the whole $\mathbb{R}$.

Q3. For $x > 0$, $x > e^{-\frac{1}{x}} > 0$ and $|\sin\cos(\frac{1}{x})| \leq 1$. Therefore

$$\lim_{x\to 0^+} e^{-\frac{1}{x}}\sin\left(\cos\frac{1}{x}\right) = 0.$$

Since we are given that $f$ is continuous, $a = f(0) = \lim_{x\to 0^+} e^{-\frac{1}{x}}\sin\left(\cos\dfrac{1}{x}\right) = 0$.

Now $\lim_{x\to 0^-} e^{e^{\sin\frac{1}{x}}}$ does not exist. So if $f$ need to be continuous, we must have $b = 0$.

Q4. Let $x \in \mathbb{R}$. Define a sequence $\{a_n\}$ by $a_n = \dfrac{x}{2^n}$. Then we have $\lim_{n\to\infty} a_n = 0$. Since $f(a_n) = f(a_{n+1})$ for all positive integer $n$ and $f$ is continuous on $\mathbb{R}$, we have

$$f(x) = f(x/2) = \lim_{n\to\infty} f(a_n) = f(0).$$

Since $x$ is arbitrary, we conclude that $f(x) = f(0)$ for all $x \in \mathbb{R}$.

Q5. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a univariate polynomial of odd degree with real coefficients and $a_n \neq 0$. If $\frac{1}{a_n}P(x)$ has a root $c$ in real numbers, then $P(c) = a_n \frac{1}{a_n} P(c) = 0$. So we may assume $a_n = 1$. Now let $M = n(|a_{n-1}| + |a_{n-2}| + \cdots + |a_0| + 1) > 0$. We have

$$(-(|a_{n-1}|+\cdots+|a_0|+1))^n \leq |a_k|(|a_{n-1}|+\cdots+|a_0|+1)^k \leq (|a_{n-1}|+|a_{n-2}|+\cdots+|a_0|+1)^n$$

for $k = 0, 1, \ldots n-1$. So $P(M) > 0$ and $P(-M) < 0$ and by applying Intermediate Value Theorem on $[-M, M]$, there is a root between $-M$ and $M$.

Q6. (a) Since $\lim_{x\to 0^+} \sqrt[2016]{x} = 0$ and $|\sin\left(2e^{\frac{1}{x}}\right)| \leq 1$, we have $\lim_{x\to 0^+} \sqrt[2016]{x}\sin\left(2e^{1/x}\right) + 1 = 1$ and $\lim_{x\to 0^-} f(x) = 1$. Therefore $f$ is continuous at 0. At other points, the continuity of $f$ follows from composting continuous functions.

(b) Since $f$ is continuous on $\mathbb{R}$, the function $g(x) := f(x) - x$ is also continuous on $\mathbb{R}$. In particular, $g$ is continuous on $[0, 1]$. Furthermore, $g(0) = f(0) - 0 = 1$, $g(1) = f(1) - 1 = \sin\left(2e^{1/1}\right) = \sin(2e)$. Since $\pi < 4 < 2e < 6 < 2\pi$, we have $g(1) < 0$. By Intermediate Value Theorem, there exists $c \in [0, 1]$ such that $g(c) = 0$, that is, $f(c) = c$.