THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 4

Properties of limit Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions such that their limit at $x = a$ exist for some $a \in \mathbb{R}$. Then we have

1. $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$.
2. $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$.
3. $\lim_{x\to a}(cf(x)) = c \lim_{x\to a} f(x)$ for some $c \in \mathbb{R}$.
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$.

Squeeze theorem Let $f(x), g(x), h(x)$ be real valued functions and $a \in \mathbb{R}$. Suppose

1. $f(x) \leq g(x) \leq h(x)$ for any $x \neq a$ in an open interval (c, d) containing a.
2. $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ for some $L \in \mathbb{R}$.

Then $\lim_{x\to a} g(x) = L$.

Continuity A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **continuous at** a if

$$\lim_{x\to a} f(x) = f(a).$$

f is said to be **continuous** if it is continuous at a for all a in its domain.
Sum, product, and composition of continuous functions are continuous.
If a function is continuous at a and its value at a is non-zero, then its reciprocal is continuous at a.
Given a sequence of real numbers $\{x_n\}$ which has a limit $a \in \mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{R}$ such that it is continuous at a, we have

$$\lim_{n\to\infty} f(x_n) = f(a)$$

Examples of continuous function $x^n, e^x, \sin x, \cos x$ are continuous on $\mathbb{R}$ for all positive integer n.
$\ln x$, $\sqrt[n]{x}$ are continuous on $(0, \infty)$ for all positive integer n.

Intermediate value theorem Suppose $f(x)$ is a continuous function on a **closed and bounded interval** $[a, b]$. Then for any real number c between $f(a)$ and $f(b)$ (exclusive), there exists $\zeta \in (a, b)$ such that $f(\zeta) = c$.

Extreme value theorem Suppose $f(x)$ is a continuous function on a **closed and bounded interval** $[a, b]$. Then there exists $\alpha, \beta \in [a, b]$ such that for any $x \in [a, b]$, we have

$$f(\alpha) \leq f(x) \leq f(\beta).$$

Problems that may be demonstrated in class :

Q1. Find the limit of the following expressions.

1. $\displaystyle\lim_{x\to 0}\frac{\ln(x^4+x^2+3)}{\ln(x^6+6x^4+9)}$

2. $\displaystyle\lim_{x\to \infty}\frac{\ln(x^4+x^2+3)}{\ln(x^6+6x^4+9)}$

3. $\displaystyle\lim_{x\to\infty} e^{-x}\tanh x$

4. $\displaystyle\lim_{x\to 0}\frac{(\ln(\sec(x)))^2}{\cos(\sin(\tan(x)))}$

5. $\displaystyle\lim_{x\to 0}\frac{\sin(e^{2x^2}-1)}{e^{2x^2}-1}$

6. $\displaystyle\lim_{x\to 0} x\cos\left(\frac{1}{x}\right)$

Q2. Determine where the function $f(x)=\begin{cases} x\sin\dfrac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$ is continuous.

Q3. Given that the function $f(x)=\begin{cases} e^{-\frac{1}{x}}\sin\left(\cos\dfrac{1}{x}\right) & \text{if } x>0\\ a & \text{if } x=0\\ be^{e^{\sin\frac{1}{x}}} & \text{if } x<0\end{cases}$ is continuous on $\mathbb{R}$ where $a,b\in\mathbb{R}$. Find the value of a and b.

Q4. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(x)=f(x/2)$ for all real number x. Show that $f(x)=f(0)$ for all real number x.

Q5. Show that every (univariate) polynomial of odd degree (with real coefficients) has a root in real numbers.

Q6. Consider $f(x)=\begin{cases} \sqrt[2016]{x}\sin\left(2e^{1/x}\right)+1 & \text{if } x>0\\ 1 & \text{if } x\leq 0\end{cases}$.

(a) Show that f is a continuous function on $\mathbb{R}$.

(b) Prove that f has a fixed point in the interval $[0,1]$. (That is, there exists $x\in[0,1]$ such that $f(x)=x$.)