THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 3 Solutions

Problems that may be demonstrated in class :

Given that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent, and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

Q1. Are the following infinite series convergent? Prove it.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n}{n^4}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{3n-2}}$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{4^n}{3^n+1}$

(e) $\displaystyle\sum_{n=2}^{\infty}\frac{n}{\ln n}$

(f) $\displaystyle\sum_{n=1}^{\infty}(-1)^n$

Q2. By using comparison test, prove the following statement: If $\displaystyle\sum_{n=1}^{\infty}a_n$ with $a_n > 0$ is convergent, then $\displaystyle\sum_{n=1}^{\infty}a_n^2$ is convergent.

Q3. (a) If $\displaystyle\sum_{n=1}^{\infty}a_n$ is absolutely convergent and (b_n) is a bounded sequence, show that $\displaystyle\sum_{n=1}^{\infty}a_nb_n$ is absolutely convergent.

(b) Give an example such that the above statement is false if *absolutely convergent* is replaced by *convergent*.

Q4. Compute the following limits:

(a) $\displaystyle\lim_{x\to1}(x+1)$

(b) $\displaystyle\lim_{x\to1}\frac{x^2-1}{x-1}$

(c) $\displaystyle\lim_{x\to0}\frac{e^x-1}{x}$

(d) $\displaystyle\lim_{x\to\infty}\frac{6e^{4x}-e^{-2x}}{8e^{5x}-e^{2x}+3e^{-x}}$

(e) $\displaystyle\lim_{x\to\infty}x-\sqrt{x^2+x}$

(f) $\lim_{x\to\infty} \frac{3x^2+7x+5}{5x^2+2}$

(g) $\lim_{x\to 0} x \sin \frac{1}{x}$

(h) $\lim_{x\to 1} \frac{x^5-1}{x-1}$

Q5. (a) Let $a \in \mathbb{R}$. Show that if $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a} [f(x)]^2$ exist.

(b) Is the converse true? Prove or disprove.

Solution Q1. (a) Note that $\frac{|\cos n|}{n^4} \leq \frac{1}{n^4} \leq \frac{1}{n^2}$ for all $n \geq 1$.

By comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent.

(b) Note that $\frac{1}{(n+1)(n+2)} \leq \frac{1}{n^2}$ for all $n \geq 1$.

By comparison test, $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$ is convergent.

(c) Since $\frac{1}{\sqrt{3n-2}} \geq \frac{1}{\sqrt{3n}} \geq \frac{1}{\sqrt{3}n}$ for all $n \geq 1$.

By comparison test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-2}}$ is divergent.

(d) Since $\lim_{n\to\infty} \frac{4^n}{3^n+1} = \infty \neq 0$, $\sum_{n=1}^{\infty} \frac{4^n}{3^n+1}$ is divergent.

(e) Since $\frac{n}{\ln n} \geq 1 \, \forall n \geq 2$, therefore $\lim_{n\to\infty} \frac{n}{\ln n} \neq 0$ and $\sum_{n=2}^{\infty} \frac{n}{\ln n}$ is divergent.

(f) There are two methods.

- Observe that

$$s_{2n} = \sum_{k=1}^{2n} (-1)^k = (-1) + 1 + (-1) + \cdots + (-1) + 1 = 0$$

$$s_{2n+1} = \sum_{k=1}^{2n+1} (-1)^k = (-1) + 1 + (-1) + \cdots + 1 + (-1) = -1$$

Therefore the sequence $\{s_n\}$ is not convergent. Therefore we have $\sum_{n=0}^{\infty} (-1)^n$ is divergent.

- Since $\lim_{n\to\infty}(-1)^n \neq 0$, therefore $\sum_{n=0}^{\infty}(-1)^n$ is divergent.

Q2. Since $\sum_{n=0}^{\infty} a_n$ converges, we have $\lim_{n\to\infty} a_n = 0$, which implies that there exists N such that

$$a_n < 1 \text{ for all } n \geq N$$

By comparison test, we have $\sum_{n=N}^{\infty} a_n^2$ converge, and so does $\sum_{n=1}^{\infty} a_n^2$

Q3. (a) By assumption, we have

(i) $\sum_{n=0}^{\infty} |a_n|$ converges

(ii) There exists M such that $|b_n| <= M$ for all $n \geq 1$

Note also that $|a_n b_n| \leq M|a_n|$ for all $n \geq 1$ and by comparison test, we have $\sum_{n=1}^{\infty} |a_n b_n|$ converges.

(b) Consider $a_n = \frac{(-1)^n}{n}$, $b_n = (-1)^n$

Q4. (a) 2

(b)

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2$$

(c) Note that for any $-1 \leq x \leq 1$

$$\frac{e^x-1}{x} = \frac{1+x+\frac{x^2}{2}+\cdots-1}{x} \leq 1+\frac{x}{2}+\left(\frac{x^2}{4}+\frac{x^2}{8}+\cdots\right) = 1+\frac{x}{2}+\frac{x^2}{2}$$

$$\frac{e^x-1}{x} = \frac{1+x+\frac{x^2}{2}+\cdots-1}{x} \geq 1+\frac{x}{2}-\left(\frac{x^2}{4}+\frac{x^2}{8}+\cdots\right) = 1+\frac{x}{2}-\frac{x^2}{2}$$

By squeeze theorem, we have $\lim_{x\to 0}\frac{e^x-1}{x} = 1$

(d)

$$\begin{aligned}\lim_{x\to\infty} \frac{6e^{4x}-e^{-2x}}{8e^{5x}-e^{2x}+3e^{-x}} &= \lim_{x\to\infty} \frac{e^{-5x}(6e^{4x}-e^{-2x})}{e^{-5x}(8e^{5x}-e^{2x}+3e^{-x})} \\ &= \lim_{x\to\infty} \frac{6e^{-x}-e^{-7x}}{8-e^{-3x}+3e^{-6x}} \\ &= 0\end{aligned}$$

(e)

$$\begin{aligned}\lim_{x\to\infty} \frac{3x^2+7x+5}{5x^2+2} &= \lim_{x\to\infty} \frac{\frac{1}{x^2}(3x^2+7x+5)}{\frac{1}{x^2}(5x^2+2)} \\ &= \lim_{x\to\infty} \frac{3+7x^{-1}+5x^{-2}}{5+2x^{-2}} \\ &= \frac{3}{5}\end{aligned}$$

(f)

$$\begin{aligned}\lim_{x\to\infty} x-\sqrt{x^2+x} &= \lim_{x\to\infty}\frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}}\\ &= \lim_{x\to\infty}\frac{x}{x+\sqrt{x^2+x}}\\ &= \lim_{x\to\infty}\frac{1}{1+\sqrt{1+\frac{1}{x}}}\\ &= \frac{1}{2}\end{aligned}$$

(g) Since $-|x| \leq x\sin\frac{1}{x} \leq |x|\ \forall x \neq 0$, and $\lim_{x\to 0}|x| = \lim_{x\to 0} -|x| = 0$, by squeeze theorem,

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0$$

(h) Note that $x^5-1=(x-1)(x^4+x^3+x^2+x+1)$

$$\lim_{x\to 1}\frac{x^5-1}{x-1} = \lim_{x\to 1}(x^4+x^3+x^2+x+1) = 5$$

Q5. (a) Assume $\lim_{x\to a} = L$. Then

$$\lim_{x\to a}[f(x)]^2 = \lim_{x\to a} f(x)f(x) = \lim_{x\to a} f(x)\lim_{x\to a} f(x) = L^2$$

(b) No. We can disprove by providing a counter-example. Consider the following function:

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$$

$\lim_{x\to 0}[f(x)]^2 = 1$ but $\lim_{x\to 0^+} f(x) = 1 \neq -1 = \lim_{x\to 0^-} f(x)$