THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 1 Solutions

Problems that may be demonstrated in class :

Q1. Show that for all real number x not equal to $\frac{n\pi}{2}$ for any integer n, we have

$$(\sin x + \cos x)(\tan x + \cot x) = (\sec x + \csc x).$$

Q2. Show that for all real number x not equal to $\frac{n\pi}{2}$ for any integer n, we have

$$\frac{\cos x}{1 \pm \sin x} = \frac{1 \mp \sin x}{\cos x}.$$

Q3. Show that for all real number x not equal to $\frac{n\pi}{2}$ for any integer n, we have

$$\sin x - \csc x = -\cot x \cos x.$$

Q4. Show that for all real number α, β, γ with $\alpha + \beta + \gamma = \pi$, we have

$$\sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

Q5. Prove that for all positive integer n, we have

$$\sum_{k=1}^{n} k^2 = 1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Q6. Prove that for all positive integer n, we have

$$\sum_{k=1}^{n} \sin k = \frac{\sin\frac{n+1}{2}}{\sin\frac{1}{2}}\sin\frac{n}{2}.$$

Q7. Prove that for all positive integer n and real number x not equal to a multiple of π, we have

$$\prod_{k=0}^{n-1} \cos(2^k x) = \cos x\cos(2x)\cos(4x)\cdots\cos(2^{n-1}x) = \frac{\sin(2^n x)}{2^n \sin x}.$$

Q8. Prove that for all positive integer n, we have

$$53^n - 46^n - 31^n + 24^n \text{ is divisible by } 77.$$

Solution Q1.

$$\begin{aligned}
\text{L.H.S.} =& (\sin x + \cos x)(\tan x + \cot x)\\
=& (\sin x + \cos x)\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)\\
=& (\sin x + \cos x)\left(\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right)\\
=& \frac{\sin x + \cos x}{\sin x \cos x}\\
=& \frac{1}{\sin x} + \frac{1}{\cos x}\\
=& \sec x + \csc x = \text{R.H.S.}
\end{aligned}$$

Q2.

$$\begin{aligned}
\text{L.H.S.} =&\frac{\cos x}{1 \pm \sin x} \\
=&\frac{\cos x}{1 \pm \sin x}\frac{1 \mp \sin x}{1 \mp \sin x} \\
=&\frac{\cos x(1 \mp \sin x}{1 - \sin^2 x} \\
=&\frac{\cos x(1 \mp \sin x}{\cos^2 x} \\
=&\frac{1 \mp \sin x}{\cos x} = \text{R.H.S.}
\end{aligned}$$

Q3.

$$\begin{aligned}
\text{L.H.S.} =&\sin x - \csc x \\
=&\frac{\sin^2 x - 1}{\sin x} \\
=&\frac{-\cos^2 x}{\sin x} \\
=&-\cot x \cos x = \text{R.H.S.}
\end{aligned}$$

Q4.

$$\begin{aligned}
\text{R.H.S.} =&4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} \\
=&4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\left(\frac{\pi}{2} - \frac{\alpha+\beta}{2}\right) \\
=&4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) \\
=&4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\left(\sin\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\beta}{2}\cos\frac{\alpha}{2}\right) \\
=&4\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\cos^2\frac{\beta}{2} + 4\cos\frac{\beta}{2}\sin\frac{\beta}{2}\cos^2\frac{\alpha}{2} \\
=&\sin\alpha\left(2\cos^2\frac{\beta}{2}\right) + \sin\beta\left(2\cos^2\frac{\alpha}{2}\right) \\
=&\sin\alpha(\cos\beta + 1) + \sin\beta(\cos\alpha + 1) \\
=&\sin\alpha + \sin\beta + (\sin\alpha\cos\beta + \sin\beta\cos\alpha) \\
=&\sin\alpha + \sin\beta + \sin(\alpha+\beta) \\
=&\sin\alpha + \sin\beta + \sin(\pi-\gamma) \\
=&\sin\alpha + \sin\beta + \sin\gamma = \text{R.H.S.}
\end{aligned}$$

Q5. Let $P(n)$ be the proposition $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

For $n = 1$, R.H.S. $= \dfrac{1(1+1)(2\cdot 1+1)}{6} = 1 =$ L.H.S.. So $P(1)$ is true.

Assume $P(\ell)$ is true for some positive integer ℓ, that is, $\displaystyle\sum_{k=1}^{\ell} k^2 = \frac{\ell(\ell+1)(2\ell+1)}{6}$.

For $n = \ell + 1$, we have

$$\begin{aligned}\text{L.H.S.} &= \sum_{k=1}^{\ell+1} k^2 \\ &= \sum_{k=1}^{\ell} k^2 + (\ell+1)^2 \\ &= \frac{\ell(\ell+1)(2\ell+1)}{6} + (\ell+1)^2 \\ &= \frac{\ell(2\ell+1)(\ell+1) + 6(\ell+1)(\ell+1)}{6} \\ &= \frac{(2\ell^2 + \ell + 6\ell + 6)(\ell+1)}{6} \\ &= \frac{(2\ell+3)(\ell+2)(\ell+1}{6} = \text{R.H.S.}\end{aligned}$$

So $P(\ell+1)$ is true if $P(\ell)$ is true.
By mathematical induction, $P(n)$ is true for all positive integer n, that is,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Q6. Let $P(n)$ be the proposition $\displaystyle\sum_{k=1}^{n} \sin k = \frac{\sin\frac{n+1}{2}}{\sin\frac{1}{2}} \sin\frac{n}{2}$.

For $n = 1$, R.H.S. $= \dfrac{\sin\frac{1+1}{2}}{\sin\frac{1}{2}} \sin\frac{1}{2} = \sin 1 =$ L.H.S.. So $P(1)$ is true.

Assume $P(\ell)$ is true for some positive integer ℓ, that is, $\displaystyle\sum_{k=1}^{\ell} \sin k = \frac{\sin\frac{\ell+1}{2}}{\sin\frac{1}{2}} \sin\frac{\ell}{2}$.

For $n = \ell + 1$, we have

$$\begin{aligned}
\text{L.H.S.} &= \sum_{k=1}^{\ell+1} \sin k \\
&= \sum_{k=1}^{\ell} \sin k + \sin(\ell+1) \\
&= \frac{\sin\frac{\ell+1}{2}}{\sin\frac{1}{2}} \sin\frac{\ell}{2} + \sin(\ell+1) \\
&= \frac{\sin\frac{\ell+1}{2}\sin\frac{\ell}{2} + \sin(\ell+1)\sin\frac{1}{2}}{\sin\frac{1}{2}} \\
&= \frac{\frac{1}{2}\left[\cos(\frac{\ell+1}{2} - \frac{\ell}{2}) - \cos(\frac{\ell+1}{2} + \frac{\ell}{2})\right] + \frac{1}{2}\left[\cos(\ell+1-\frac{1}{2}) - \cos(\ell+1+\frac{1}{2})\right]}{\sin\frac{1}{2}} \\
&= \frac{\frac{1}{2}\left[\cos\frac{1}{2} - \cos(\ell+\frac{3}{2})\right]}{\sin\frac{1}{2}} \\
&= \frac{-\left[\sin(\frac{1}{2}(\frac{1}{2}+\ell+\frac{3}{2}))\sin(\frac{1}{2}(\frac{1}{2}-(\ell+\frac{3}{2})))\right]}{\sin\frac{1}{2}} \\
&= \frac{\sin\frac{\ell+2}{2}}{\sin\frac{1}{2}}\sin\frac{\ell+1}{2} = \text{R.H.S.}
\end{aligned}$$

So $P(\ell+1)$ is true if $P(\ell)$ is true.
By mathematical induction, $P(n)$ is true for all positive integer n, that is,

$$\sum_{k=1}^{n} \sin k = \frac{\sin\frac{n+1}{2}}{\sin\frac{1}{2}} \sin\frac{n}{2}.$$

Q7. Let $P(n)$ be the proposition $\displaystyle\prod_{k=0}^{n-1} \cos(2^k x) = \frac{\sin(2^n x)}{2^n \sin x}$.

For $n = 1$, R.H.S. $= \frac{\sin(2x)}{2\sin x} = \frac{2\sin x\cos x}{2\sin x} = \cos x$. So $P(1)$ is true.

Assume $P(\ell)$ is true for some positive integer ℓ, that is, $\displaystyle\prod_{k=0}^{\ell-1} \cos(2^k x) = \frac{\sin(2^\ell x)}{2^\ell \sin x}$.

For $n = \ell + 1$, we have

$$\begin{aligned}
\text{L.H.S.} &= \prod_{k=0}^{(\ell+1)-1} \cos(2^k x) \\
&= \cos(2^\ell x) \prod_{k=0}^{\ell-1} \cos(2^k x) \\
&= \cos(2^\ell x) \frac{\sin(2^\ell x)}{2^\ell \sin x} \\
&= \frac{\sin(2^\ell x)\cos(2^\ell x)}{2^\ell \sin x} \\
&= \frac{\sin(2^{(\ell+1)} x)}{2^{\ell+1}\sin x} = \text{R.H.S.}
\end{aligned}$$

So $P(\ell+1)$ is true if $P(\ell)$ is true.
By mathematical induction, $P(n)$ is true for all positive integer n, that is,

$$\prod_{k=0}^{n-1} \cos(2^k x) = \frac{\sin(2^n x)}{2^n \sin x}.$$

Q8. Let $P(n)$ be the proposition $53^n - 46^n - 31^n + 24^n$ is divisible by 77.
For $n = 1$, $53^1 - 46^1 - 31^1 + 24^1 = 0 = 0 \cdot 77$. So $P(1)$ is true.
For $n = 2$, $53^2 - 46^2 - 31^2 + 24^2 = 308 = 4 \cdot 77$. So $P(2)$ is true.
Assume $P(n)$ and $P(n+1)$ is true for some positive integer n, that is, there exists integer k_n, k_{n+1} such that $53^n - 46^n - 31^n + 24^n = 77k_n$ and $53^{n+1} - 46^{n+1} - 31^{n+1} + 24^{n+1} = 77k_{n+1}$.
Notice that $53 \cdot 24 - 46 \cdot 31 = -154 = -2 \cdot 77$ For the case $n+2$, we have

$$\begin{aligned} 53^{n+2} - 46^{n+2} - 31^{n+2} + 24^{n+2} =& (53+24)(53^{n+1} - 46^{n+1} - 31^{n+1} + 24^{n+1}) \\ & - 31 \cdot 46(53^n - 46^n - 31^n + 24^n) \\ & + 46^{n+1}(46 - 53 - 24 + 31) + 31^{n+1}(31 - 53 - 24 + 46) \\ & - 53^n(53 \cdot 24 - 31 \cdot 46) - 24^n(53 \cdot 24 - 31 \cdot 46) \\ =& 77(77k_{n+1}) - 1426 \cdot 77k_n + 46^{n+1}(0) + 31^{n+1}(0) \\ & - 53^n(-2 \cdot 77) - 24^n(-2 \cdot 77) \\ =& 77(77k_{n+1} - 1426k_n + 2 \cdot 53^n + 2 \cdot 24^n) \end{aligned}$$

So $53^{n+2} - 46^{n+2} - 31^{n+2} + 24^{n+2}$ is divisible by 77 if $P(n)$ and $P(n+1)$ is true.
By mathematical induction, $P(n)$ is true for all positive integer n, that is,

$$53^n - 46^n - 31^n + 24^n \text{ is divisible by 77.}$$