THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 11 Solutions

Problems that may be demonstrated in class :

Q1. Suppose a, b are real constants. Let $I_n = \int \frac{dx}{((x-a)^2+b^2)^n}$ for any positive integer n. Prove the following recursive relation: $I_1 = \frac{1}{b}\arctan\left(\frac{x-a}{b}\right) + C$ and

$$I_n = \frac{2n-3}{2b^2(n-1)} I_{n-1} + \frac{x-a}{2b^2(n-1)((x-a)^2+b^2)^{n-1}} \quad \forall n > 1.$$

Q2. Compute the following indefinite/definite integrals:

(a) $\int \frac{x^2+x+1}{\sqrt{x^2-9}}dx$; (b) $\int \sqrt{x^2-4}dx$; (c) $\int \frac{dx}{(x^2+1)^{3/2}}$;

(d) $\int \frac{dx}{x^4+8x^2+16}$; (e) $\int_{-1/2}^{0} \frac{x^6 dx}{(x-2)(x^2+2x-3)}$; (f) $\int \frac{(x^3-7x)dx}{(x-5)^4}$;

(g) $\int_0^{\pi/2} \frac{-\sin x+4\cos x+2}{3\cos x+5}dx$; (h) $\int_{-\pi/4}^{\pi/4} \frac{7\sin x-6\cos x+5\tan x}{3\cos x+2}dx$; (i) $\int \frac{\sin x-1}{\sin x+\tan x}dx$.

Q3. As a variant of t-substitution, for an integral $\int R(\sin^2 x, \cos^2 x)dx$, where R is a rational function, we can make substitution $t = \tan x$. Use this method to evaluate:

(a) $\int_0^{\pi/4} \frac{\sin^2 x-2}{\cos^2 x+1}dx$; (b) $\int_{-\pi/4}^{\pi/4} \frac{dx}{1-\sin^4 x}$; (c) $\int_{-\pi/4}^{\pi/4} \frac{dx}{\tan^2 x+\sec^2 x}$.

Q4. Let a, b be non-zero real constants. Prove that for any positive integer n,

$$\int \frac{dx}{(ae^x+b)^n} = \frac{x}{b^n} - \frac{1}{b^n}\ln|ae^x+b| + \sum_{m=1}^{n-1} \frac{1}{mb^{n-m}(ae^x+b)^m} + C,$$

where empty sum equals zero.

Solutions :

Q1. Let $x = a + b\tan\theta$. Then $dx = b\sec^2\theta d\theta$.

$$I_1 = \int \frac{b\sec^2\theta d\theta}{(b\tan\theta)^2+b^2} = \int \frac{b\sec^2\theta d\theta}{b^2\sec^2\theta} = \int \frac{d\theta}{b} = \frac{\theta}{b} + C = \frac{1}{b}\arctan\left(\frac{x-a}{b}\right) + C.$$

Suppose $n \in \mathbb{N}$ and $n > 1$. By integration by parts,

$$\begin{aligned}
I_{n-1} &= \frac{x}{((x-a)^2+b^2)^{n-1}} - \int x d\left(\frac{1}{((x-a)^2+b^2)^{n-1}}\right) \\
&= \frac{x}{((x-a)^2+b^2)^{n-1}} + 2(n-1)\int \frac{x(x-a)}{((x-a)^2+b^2)^n}dx \\
&= \frac{x}{((x-a)^2+b^2)^{n-1}} + 2(n-1)\int \frac{[(x-a)^2+b^2]+a(x-a)-b^2}{((x-a)^2+b^2)^n}dx \\
&= \frac{x}{((x-a)^2+b^2)^{n-1}} + 2(n-1)I_{n-1} + \int \frac{2a(n-1)(x-a)}{((x-a)^2+b^2)^n}dx - 2(n-1)b^2 I_n \\
&= \frac{x}{((x-a)^2+b^2)^{n-1}} + 2(n-1)I_{n-1} - \frac{a}{((x-a)^2+b^2)^{n-1}} - 2b^2(n-1)I_n \\
&= \frac{x-a}{((x-a)^2+b^2)^{n-1}} + 2(n-1)I_{n-1} - 2b^2(n-1)I_n.
\end{aligned}$$

Rearranging terms, we get

$$\begin{aligned}
I_n &= \frac{1}{2b^2(n-1)}\left(\frac{x-a}{((x-a)^2+b^2)^{n-1}} + 2(n-1)I_{n-1} - I_{n-1}\right) \\
&= \frac{2n-3}{2b^2(n-1)}I_{n-1} + \frac{x-a}{2b^2(n-1)((x-a)^2+b^2)^{n-1}}.
\end{aligned}$$

Q2. (a) Let $x = 3\sin\theta$, $-\pi/2 \le \theta \le \pi/2$. Then $dx = 3\cos\theta d\theta$.

$$\begin{aligned}
\int \frac{x^2+x+1}{\sqrt{9-x^2}}dx &= \int \frac{9\sin^2\theta + 3\sin\theta + 1}{3\cos\theta} \cdot 3\cos\theta d\theta \\
&= \int \left(\frac{9}{2}(1-\cos 2\theta) + 3\sin\theta + 1\right) d\theta \\
&= \int \left(-\frac{9}{2}\cos 2\theta + 3\sin\theta + \frac{11}{2}\right) d\theta \\
&= -\frac{9}{4}\sin 2\theta - 3\cos\theta + \frac{11\theta}{2} + C \\
&= -\frac{9}{2}\sin\theta\cos\theta - 3\cos\theta + \frac{11\theta}{2} + C \\
&= -\frac{1}{2}x\sqrt{9-x^2} - \sqrt{9-x^2} + \frac{11}{2}\arcsin\left(\frac{x}{3}\right) + C.
\end{aligned}$$

(b) Let $x = 2\sec\theta$, $0 \le \theta < \pi/2$. Then $dx = 2\tan\theta\sec\theta d\theta$.

$$\begin{aligned}
\int \tan^2\theta\sec\theta d\theta = \int \tan\theta d(\sec\theta) &= \tan\theta\sec\theta - \int \sec\theta d\tan\theta \\
&= \tan\theta\sec\theta - \int \sec^3\theta d\theta \\
&= \tan\theta\sec\theta - \int \tan^2\theta\sec\theta d\theta - \int \sec\theta d\theta \\
&= \frac{\tan\theta\sec\theta}{2} - \frac{1}{2}\int \sec\theta d\theta \\
&= \frac{\tan\theta\sec\theta}{2} - \frac{1}{2}\ln|\tan\theta + \sec\theta| + C.
\end{aligned}$$

Then

$$\begin{aligned}
\int \sqrt{x^2-4}dx = \int 2\tan\theta \cdot 2\tan\theta\sec\theta d\theta &= 4\int \tan^4\theta\sec\theta d\theta \\
&= 2\tan\theta\sec\theta - 2\ln|\tan\theta + \sec\theta| + C \\
&= \frac{x\sqrt{x^2-4}}{2} - 2\ln|x + \sqrt{x^2-4}| + C.
\end{aligned}$$

Alternative method:

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2-4}} = \int \frac{2\tan\theta\sec\theta d\theta}{2\tan\theta} &= \int \sec\theta d\theta = \ln|\tan\theta + \sec\theta| + C' \\
&= \ln\left|\frac{\sqrt{x^2-4}+x}{2}\right| + C' = \ln|x + \sqrt{x^2-4}| - \ln 2 + C' \\
&= \ln|x + \sqrt{x^2-4}| + C.
\end{aligned}$$

Now we perform integration by parts:

$$\begin{aligned}
\int \sqrt{x^2-4}dx &= x\sqrt{x^2-4} - \int \frac{x^2dx}{\sqrt{x^2-4}} \\
&= x\sqrt{x^2-4} - \int \sqrt{x^2-4}dx - 4\int \frac{dx}{\sqrt{x^2-4}} \\
&= \frac{x\sqrt{x^2-4}}{2} - 2\int \frac{dx}{\sqrt{x^2-4}} \\
&= \frac{x\sqrt{x^2-4}}{2} - 2\ln|x+\sqrt{x^2-4}| + C.
\end{aligned}$$

(c) Let $x = \tan\theta$, $-\pi/2 < \theta < \pi/2$. Then $dx = \sec^2\theta d\theta$.

$$\int \frac{dx}{(x^2+1)^{3/2}} = \int \frac{\sec^2\theta d\theta}{\sec^3\theta} = \int \cos\theta d\theta = -\sin\theta + C = -\frac{x}{\sqrt{x^2+1}} + C.$$

(d) Note that $\int \frac{dx}{x^4+8x^2+16} = \int \frac{dx}{(x^2+4)^2}$. By integration by parts,

$$\begin{aligned}
\int \frac{dx}{x^2+4} &= \frac{x}{x^2+4} + 2\int \frac{x^2dx}{(x^2+4)^2} \\
&= \frac{x}{x^2+4} + 2\int \frac{(x^2+4)dx}{(x^2+4)^2} - 8\int \frac{dx}{(x^2+4)^2} \\
&= \frac{x}{x^2+4} + 2\int \frac{dx}{x^2+4} - 8\int \frac{dx}{(x^2+4)^2}, \\
\int \frac{dx}{x^4+8x^2+16} &= \int \frac{dx}{(x^2+4)^2} = \frac{1}{8}\left(\frac{x}{x^2+4} + \int \frac{dx}{x^2+4}\right) \\
&= \frac{1}{8}\left(\frac{x}{x^2+4} + \frac{1}{2}\arctan\frac{x}{2}\right) + C.
\end{aligned}$$

(e) Note that $(x-2)(x^2+2x-3) = x^3-7x+6 = (x-1)(x-2)(x+3)$. Perform long division:

$$\begin{array}{r|l}
 & x^3 \qquad\qquad + 7x \; - 6 \\ \hline
x^3-7x+6 & x^6 \\
 & -x^6 + 7x^4 - 6x^3 \\ \hline
 & 7x^4 - 6x^3 \\
 & -7x^4 \qquad + 49x^2 - 42x \\ \hline
 & -6x^3 + 49x^2 - 42x \\
 & 6x^3 \qquad - 42x + 36 \\ \hline
 & 49x^2 - 84x + 36
\end{array}$$

By partial fraction decomposition,

$$\begin{aligned}
&\frac{49x^2-84x+36}{(x-1)(x-2)(x+3)} \\
=&\frac{49(1)^2-84(1)+36}{(x-1)(1-2)(1+3)} + \frac{49(2)^2-84(2)+36}{(2-1)(x-2)(2+3)} + \frac{49(-3)^2-84(-3)+36}{(-3-1)(-3-2)(x+3)} \\
=&-\frac{1}{4(x-1)} + \frac{64}{5(x-2)} + \frac{729}{20(x+3)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{-1/2}^{0} \frac{x^6 dx}{(x-2)(x^2+2x-3)} \\
&= \int_{-1/2}^{0} \left(x^3 + 7x - 6 - \frac{1}{4(x-1)} + \frac{64}{5(x-2)} + \frac{729}{20(x+3)} \right) dx \\
&= \left[\frac{x^4}{4} + \frac{7x^2}{2} - 6x - \frac{1}{4} \ln|x-1| + \frac{64}{5} \ln|x-2| + \frac{729}{20} \ln|x+3| \right]_{-1/2}^{0} \\
&= \left(\frac{64}{5} \ln 2 + \frac{729}{20} \ln 3 \right) - \left(\frac{1}{64} + \frac{7}{8} + 3 - \frac{1}{4} \ln \frac{3}{2} + \frac{64}{5} \ln \frac{5}{2} + \frac{729}{20} \ln \frac{5}{2} \right) \\
&= -\frac{249}{64} + \frac{309}{5} \ln 2 + \frac{367}{10} \ln 3 - \frac{197}{4} \ln 5.
\end{aligned}$$

(f) Let $u = x - 5$. Then $du = dx$.

$$\begin{aligned}
\int \frac{(x^3 - 7x)dx}{(x-5)^4} &= \int \frac{(u+5)^3 - 7(u+5)}{u^4} du \\
&= \int \frac{u^3 + 3(5)u^2 + 3(5)^2 u + 5^3 - 7u - 35}{u^4} du \\
&= \int (u^{-1} + 15u^{-2} + 68u^{-3} + 90u^{-4}) du \\
&= \ln|u| - \frac{15}{u} - \frac{34}{u^2} - \frac{30}{u^3} + C. \\
&= \ln|x-5| - \frac{15}{x-5} - \frac{34}{(x-5)^2} - \frac{30}{(x-5)^3} + C.
\end{aligned}$$

Alternative method: let a, b, c, d be real constants such that

$$\frac{x^3 - 7x}{(x-5)^4} \equiv \frac{a}{x-5} + \frac{b}{(x-5)^2} + \frac{c}{(x-5)^3} + \frac{d}{(x-5)^4}.$$

Let $f(x) = x^3 - 7$. Then

$$\begin{aligned}
f(x) &\equiv x^3 - 7x \equiv a(x-5)^3 + b(x-5)^2 + c(x-5) + d, && (1) \\
f'(x) &\equiv 3x^2 - 7 \equiv 3a(x-5)^2 + 2b(x-5) + c, && (2) \\
f''(x) &\equiv 6x \equiv (3)(2)a(x-5) + 2!b, && (3) \\
f'''(x) &\equiv 6 \equiv 3!a. && (4)
\end{aligned}$$

Putting $x = 5$ into (1), (2), (3), (4) respectively, we get

$$a = \frac{f'''(5)}{3!} = 1; \quad b = \frac{f''(5)}{2!} = 15; \quad c = \frac{f'(5)}{1!} = 68; \quad d = \frac{f(5)}{0!} = 90.$$

Hence

$$\begin{aligned}
\int \frac{(x^3 - 7x)dx}{(x-5)^4} &= \int \left(\frac{1}{x-5} + \frac{15}{(x-5)^2} + \frac{68}{(x-5)^2} + \frac{90}{(x-5)^4} \right) dx \\
&= \ln|x-5| - \frac{15}{x-5} - \frac{34}{(x-5)^2} - \frac{30}{(x-5)^3} + C.
\end{aligned}$$

(g) Let m, n, k be the real constants such that

$$\frac{-\sin x + 4\cos x + 2}{3\cos x + 5} \equiv m + n \cdot \frac{-3\sin x}{3\cos x + 5} + \frac{k}{3\cos x + 5}.$$

Then

$$\frac{-\sin x + 4\cos x + 2}{3\cos x + 5} \equiv \frac{-3n\sin x + 3m\cos x + 5m + k}{3\cos x + 5}.$$

By comparing coefficients, $m = 4/3$, $n = 1/3$ and $k = -14/3$. Let $t = \tan\frac{x}{2}$. Then $dx = \frac{2dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$. When $x = 0$, $t = 0$; when $x = \pi/2$, $t = 1$. Hence,

$$\begin{aligned}
&\int_0^{\pi/2} \frac{-\sin x + 4\cos x + 2}{3\cos x + 5} dx \\
=&\frac{4}{3}\int_0^{\pi/2} dx + \frac{1}{3}\int_0^{\pi/2} \frac{-3\sin x}{3\cos x + 5} dx - \frac{14}{3}\int_0^{\pi/2} \frac{dx}{3\cos x + 5} \\
=&\frac{2\pi}{3} + \frac{1}{3}\left[\ln|3\cos x + 5|\right]_0^{\pi/2} - \frac{14}{3}\int_0^1 \frac{\frac{2dt}{1+t^2}}{3\frac{1-t^2}{1+t^2} + 5} \\
=&\frac{2\pi}{3} - \frac{1}{3}\ln\frac{5}{8} - \frac{14}{3}\int_0^1 \frac{dt}{t^2+4} = \frac{2\pi}{3} + \frac{1}{3}\ln\frac{5}{8} - \left[\frac{7}{3}\arctan\frac{t}{2}\right]_0^1 \\
=&\frac{2\pi}{3} + \frac{1}{3}\ln\frac{5}{8} - \frac{7}{3}\arctan\frac{1}{2}.
\end{aligned}$$

Alternative method:

$$\begin{aligned}
\int_0^{\pi/2} \frac{-\sin x + 4\cos x + 2}{3\cos x + 5} dx &= \int_0^1 \frac{-\frac{2t}{1+t^2} + 4\cdot\frac{1-t^2}{1+t^2} + 2}{3\cdot\frac{1-t^2}{1+t^2} + 5} \frac{2dt}{1+t^2} \\
&= \int_0^1 \frac{-2t^2 - 2t + 6}{(t^2+4)(t^2+1)} dt.
\end{aligned}$$

Let A, B, C, D be real constants such that

$$\frac{-2t^2 - 2t + 6}{(t^2+4)(t^2+1)} \equiv \frac{At+B}{t^2+1} + \frac{Ct+D}{t^2+4}.$$

Then

$$\begin{aligned}
-2t^2 - 2t + 6 &\equiv (At+B)(t^2+4) + (Ct+D)(t^2+1) \\
&\equiv (A+C)t^3 + (B+D)t^2 + (4A+C)t + 4B + D.
\end{aligned}$$

We have

$$\begin{cases}
A + C &= 0, \\
B + D &= -2, \\
4A + C &= -2, \\
4B + D &= 6.
\end{cases}$$

On solving, $A=-\frac{2}{3}$, $B=\frac{8}{3}$, $C=\frac{2}{3}$, $D=-\frac{14}{3}$.

$$\begin{aligned}
&\int_0^{\pi/2}\frac{-\sin x+4\cos x+2}{3\cos x+5}dx\\
=&\frac{1}{3}\int_0^1\left[\frac{-2t+8}{t^2+1}+\frac{2t-14}{t^2+4}\right]dt\\
=&\frac{1}{3}\left[-\ln(t^2+1)+8\arctan t+\ln(t^2+4)-\frac{7}{3}\arctan\frac{t}{2}\right]_0^1\\
=&\frac{1}{3}\left(-\ln 2+2\pi+\ln 5-\frac{7}{3}\arctan\frac{1}{2}-\ln 4\right)=\frac{2\pi}{3}+\frac{1}{3}\ln\frac{5}{8}-\frac{7}{3}\arctan\frac{1}{2}.
\end{aligned}$$

(h) Let $t=\tan\frac{x}{2}$. Then $dx=\frac{2dt}{1+t^2}$ and $\cos x=\frac{1-t^2}{1+t^2}$. When $x=0$, $t=0$; when $x=\frac{\pi}{4}$, $t=\tan\frac{\pi}{8}=\sqrt{2}-1$ (Why?).

$$\begin{aligned}
&\int_{-\pi/4}^{\pi/4}\frac{7\sin x-6\cos x+5\tan x}{3\cos x+2}dx\\
=&\int_0^{\pi/4}\frac{-12\cos x}{3\cos x+2}dx\quad\left(\because \frac{7\sin x+5\tan x}{3\cos x+2}\text{ is odd and }\frac{\cos x}{3\cos x+2}\text{ is even}\right)\\
=&\int_0^{\pi/4}\frac{-12\cos x-8}{3\cos x+2}dx+8\int_0^{\pi/4}\frac{dx}{3\cos x+2}\\
=&-4\int_0^{\pi/4}dx+8\int_0^{\sqrt{2}-1}\frac{\frac{2dt}{1+t^2}}{3\cdot\frac{1-t^2}{1+t^2}+2}=-\pi+16\int_0^{\sqrt{2}-1}\frac{dt}{5-t^2}\\
=&-\pi+16\int_0^{\sqrt{2}-1}\frac{dt}{(\sqrt{5}-t)(\sqrt{5}+t)}\\
=&-\pi+\frac{8}{\sqrt{5}}\int_0^{\sqrt{2}-1}\left(\frac{1}{\sqrt{5}-t}+\frac{1}{\sqrt{5}+t}\right)dt\\
=&-\pi+\frac{8}{\sqrt{5}}\left[\ln\left|\frac{\sqrt{5}+t}{\sqrt{5}-t}\right|\right]_0^{\sqrt{2}-1}=-2\pi+\frac{8}{\sqrt{5}}\ln\left|\frac{\sqrt{5}+\sqrt{2}-1}{\sqrt{5}-\sqrt{2}+1}\right|\\
=&-\pi+\frac{8\sqrt{5}}{5}\ln[(\sqrt{5}-2)(3+\sqrt{10})].
\end{aligned}$$

Remark: Why $\tan\frac{\pi}{8}=\sqrt{2}-1$? Using the following double-angle formula,

$$\begin{aligned}
1=\tan\frac{\pi}{4}=\tan\left(2\cdot\frac{\pi}{8}\right)&=\frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}},\\
\therefore \tan^2\frac{\pi}{8}+2\tan\frac{\pi}{8}-1&=0,\\
\left(\tan\frac{\pi}{8}+1\right)^2&=2,\\
\tan\frac{\pi}{8}&=\sqrt{2}-1\text{ or }-\sqrt{2}-1\text{ (rejected)}.
\end{aligned}$$

(i) Let $t = \tan\frac{x}{2}$. Then $dx = \frac{2dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$ and $\tan x = \frac{2t}{1-t^2}$.

$$\begin{aligned}
\int \frac{\sin x dx}{\sin x + \tan x} &= \int \frac{\frac{2t}{1+t^2}\cdot\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}+\frac{2t}{1-t^2}} = \int \frac{4t(1-t^2)}{2t(1-t^2)+2t(1+t^2)}\cdot\frac{dt}{1+t^2} \\
&= \int \frac{1-t^2}{1+t^2}dt = \int \frac{-(t^2+1)+2}{t^2+1}dt = -\int dt + 2\int\frac{2dt}{1+t^2} \\
&= -\int dt + \int dx = -t + x + C = x - \tan\frac{x}{2} + C \\
&= x - \frac{\sin x}{1+\cos x} + C.
\end{aligned}$$

Alternative method:

$$\begin{aligned}
\int \frac{\sin x dx}{\sin x + \tan x} &= \int \frac{dx}{1+\sec x} = \int\frac{\cos x dx}{\cos x + 1} = \int \frac{\cos x + 1 - 1}{\cos x + 1}dx \\
&= \int dx - \int \frac{dx}{1+\cos x} = x - \int \frac{dx}{2\cos^2\frac{x}{2}} \\
&= x - \int \sec^2\frac{x}{2} d\left(\frac{x}{2}\right) = x - \tan\frac{x}{2} + C = x - \frac{\sin x}{1+\cos x} + C.
\end{aligned}$$

Q3. Note that $dt = \sec^2 x dx = (1+\tan^2 x)dx = (1+t^2)dx$. Then $dx = \frac{dt}{1+t^2}$. Also, $\cos^2 x = \frac{1}{\sec^2 x} = \frac{1}{1+t^2}$ and $\sin^2 x = 1 - \cos^2 x = 1 - \frac{1}{1+t^2} = \frac{t^2}{1+t^2}$. When $x = 0$, $t = 0$; when $x = \pi/4$, $t = 1$.

(a)

$$\begin{aligned}
\int_0^{\pi/4} \frac{\sin^2 x - 2}{\cos^2 x + 1}dx &= \int_0^1 \frac{\frac{t^2}{1+t^2} - 2}{\frac{1}{1+t^2}+1}\cdot\frac{dt}{1+t^2} = \int_0^1 \frac{t^2 - 2(1+t^2)}{1+(1+t^2)}\cdot\frac{dt}{1+t^2} \\
&= -\int_0^1 \frac{t^2+2}{(t^2+2)(t^2+1)}dt = -\int_0^1\frac{dt}{t^2+1} \\
&= -\left[\arctan t\right]_0^1 = -\frac{\pi}{4}.
\end{aligned}$$

(b) Note that $\frac{1}{1-\sin^4 x}$ is an even function in x.

$$\begin{aligned}
&\int_{-\pi/4}^{\pi/4} \frac{dx}{1-\sin^4 x} \\
=&2\int_0^{\pi/4} \frac{dx}{(1-\sin^2 x)(1+\sin^2 x)} = \int_0^{\pi/4}\left(\frac{1}{1-\sin^2 x} + \frac{1}{1+\sin^2 x}\right)dx \\
=&\int_0^{\pi/4} \sec^2 x dx + \int_0^1 \frac{\frac{dt}{1+t^2}}{1+\frac{t^2}{1+t^2}} = \left[\tan x\right]_0^{\pi/4} + \frac{1}{2}\int_0^1 \frac{dt}{t^2+\frac{1}{2}} \\
=&1 + \frac{1}{2}\left[\sqrt{2}\arctan\sqrt{2}t\right]_0^1 = \frac{2+\sqrt{2}\arctan\sqrt{2}}{2}.
\end{aligned}$$

(c) Note that $\frac{1}{\tan^2 x + \sec^2 x}$ is an even function in x.

$$\begin{aligned}
\int_{-\pi/4}^{\pi/4} \frac{dx}{\tan^2 x + \sec^2 x} &= 2\int_0^{\pi/4}\frac{\cos^2 x dx}{\sin^2 x + 1} = 2\int_0^1 \frac{\frac{1}{1+t^2}\cdot\frac{dt}{1+t^2}}{\frac{t^2}{1+t^2}+1} \\
&= \int_0^1 \frac{2dt}{(t^2+1)(2t^2+1)}.
\end{aligned}$$

Let A, B, C, D be real constants such that

$$\frac{2}{(t^2+1)(2t^2+1)} \equiv \frac{At+B}{t^2+1} + \frac{Ct+D}{2t^2+1}.$$

Then

$$\begin{aligned} 2 &\equiv (At+B)(2t^2+1) + (Ct+D)(t^2+1) \\ &\equiv (2A+C)t^3 + (2B+D)t^2 + (A+C)t + B + D. \end{aligned}$$

We have

$$\begin{cases} 2A+C &= 0, \\ 2B+D &= 0, \\ A+C &= 0, \\ B+D &= 2. \end{cases}$$

On solving, $A = 0,\ B = -2,\ C = 0,\ D = 4$. Hence,

$$\begin{aligned} \int_{-\pi/4}^{\pi/4} \frac{dx}{\tan^2 x + \sec^2 x} &= 2\left(\int_0^1 \frac{dt}{t^2+\frac{1}{2}} - \int_0^1 \frac{dt}{t^2+1}\right) \\ &= 2\left(\left[\sqrt{2}\arctan\sqrt{2}t\right]_0^1 - [\arctan t]_0^1\right) \\ &= 2\left(\sqrt{2}\arctan\sqrt{2} - \frac{\pi}{4}\right). \end{aligned}$$

Remark. The reason why this is a variant of t-substitution is that both $\sin^2 x$ and $\cos^2 x$ can be expressed as a polynomial in $\cos y$:

$$\sin^2 x = \frac{1}{2}(1-\cos y) \quad \text{and} \quad \cos^2 x = \frac{1}{2}(1+\cos y),$$

where $y = 2x$. Then $R(\sin^2 x, \cos^2 x)$ is a rational function in $\cos y$ and we use t-substitution: $t = \tan\frac{y}{2} = \tan x$.

Q4. Let $I_n = \int \frac{dx}{(ae^x+b)^n}$ for any positive integer n. First of all,

$$I_1 = \frac{1}{b}\int \frac{ae^x+b-ae^x}{ae^x+b}dx = \frac{1}{b}\left(\int dx - \int \frac{ae^x dx}{ae^x+b}\right) = \frac{x}{b} - \frac{1}{b}\ln|ae^x+b| + C.$$

Assume

$$I_k = \frac{x}{b^k} - \frac{1}{b^k}\ln|ae^x+b| + \sum_{m=1}^{k-1} \frac{1}{mb^{k-m}(ae^x+b)^m} + C$$

for some positive integer k. Then

$$\begin{aligned} I_{k+1} &= \frac{1}{b}\int \frac{ae^x+b-ae^x}{(ae^x+b)^{k+1}}dx = \frac{1}{b}\left(I_k - \int \frac{ae^x dx}{(ae^x+b)^{k+1}}\right) \\ &= \frac{I_k}{b} + \frac{1}{kb(ae^x+b)^k} \\ &= \frac{x}{b^{k+1}} - \frac{1}{b^{k+1}}\ln|ae^x+b| + \sum_{m=1}^{k-1} \frac{1}{mb^{k+1-m}(ae^x+b)^m} + \frac{1}{kb(ae^x+b)^k} + C \\ &= \frac{x}{b^{k+1}} - \frac{1}{b^{k+1}}\ln|ae^x+b| + \sum_{m=1}^{k} \frac{1}{mb^{k+1-m}(ae^x+b)^m} + C. \end{aligned}$$

By mathematical induction, for any positive integer n,

$$I_n = \frac{x}{b^n} - \frac{1}{b^n} \ln|ae^x + b| + \sum_{m=1}^{n-1} \frac{1}{mb^{n-m}(ae^x + b)^m} + C.$$