THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 10

Techniques of Integration :

Integration by parts: If $u(x)$ and $v(x)$ are differentiable, then

$$\int uv'dx = uv - \int u'vdx.$$

If $u(x)$ is n-time differentiable and $v(x)$ is differentiable, then

$$\int uv'dx = \left[\sum_{r=0}^{n-1}(-1)^r u^{(r)}v_r\right] + (-1)^n \int u^{(n)}v_{n-1}dx,$$

where v_{r+1} is a primitive of v_r for any non-negative integer r with $v_0 = v$.

Reduction formula: For $I_n = \int f_n(x)dx$, express I_n in terms of I_k's with $k < n$.

Integration of products of two common functions:

1. $I = \int x^n e^{ax}dx$, where $n \in \mathbb{Z}$, $n \geq 0$ and $a \in \mathbb{R}$.

a	Method	$I =$
$= 0$	Direct integration	$\frac{x^{n+1}}{n+1} + C$
$\neq 0$	By parts: $u = x^n$, $v' = e^{ax}$	$\frac{e^{ax}}{a}\sum_{r=0}^{n} P_r^n(-\frac{1}{a})^r x^{n-r} + C$

2. $I = \int x^a(\ln x)^n dx$, where $n \in \mathbb{Z}$, $n \geq 0$ and $a \in \mathbb{R}$.

a	Method	$I =$
$= -1$	Sub.: $y = \ln x$	$\frac{(\ln x)^{n+1}}{n+1} + C$
$\neq -1$	By parts: $u = (\ln x)^n$, $v' = x^a$	$\frac{x^{a+1}}{a+1}\sum_{r=0}^{n} P_r^n(-\frac{1}{a+1})^r(\ln x)^{n-r} + C$

3. $I = \int x^n f(\alpha x)dx$, where $f(x) = \sin x$ or $\cos x$, $n \in \mathbb{Z}$, $n \geq 0$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

$f(x) =$	Method	$I =$
$\sin x$	By parts: $u = x^n$, $v' = f(\alpha x)$	$-\sum_{r=0}^{n} P_r^n \frac{x^{n-r}}{\alpha^{r+1}}\cos(\alpha x + \frac{r\pi}{2}) + C$
$\cos x$	By parts: $u = x^n$, $v' = f(\alpha x)$	$\sum_{r=0}^{n} P_r^n \frac{x^{n-r}}{\alpha^{r+1}}\sin(\alpha x + \frac{r\pi}{2}) + C$

4. $I = \int \sin\alpha x\cos\beta x dx$, $\int \sin\alpha x \sin\beta x dx$ or $\int \cos\alpha x\cos\beta x dx$, where $\alpha, \beta \in \mathbb{R}$.
Method: by trigonometric product-to-sum formulae:
   (a) $\sin\alpha x\cos\beta x = \frac{1}{2}(\sin(\alpha+\beta)x + \sin(\alpha-\beta)x)$.
   (b) $\sin\alpha x\sin\beta x = -\frac{1}{2}(\cos(\alpha+\beta)x - \cos(\alpha-\beta)x)$.
   (c) $\cos\alpha x\cos\beta x = \frac{1}{2}(\cos(\alpha+\beta)x + \cos(\alpha-\beta)x)$.

5. $I_{m,n}(\alpha) = \int \sin^m \alpha x \cos^n \alpha x dx$, where $m, n \in \mathbb{Z}$, $m, n \geq 0$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

(m, n)	Method	$I_{m,n}(\alpha) =$
(o, $\cdot$)	Sub.: $u = \cos \alpha x$	$\frac{1}{\alpha} \sum_{r=0}^{(m-1)/2} C_r^{(m-1)/2} \frac{(-1)^r \cos^{n+2r+1} \alpha x}{n+2r+1} + C$
($\cdot$, o)	Sub.: $u = \sin \alpha x$	$\frac{1}{\alpha} \sum_{r=0}^{(n-1)/2} C_r^{(n-1)/2} \frac{(-1)^r \sin^{m+2r+1} \alpha x}{m+2r+1} + C$

where 'o' means 'odd' while 'e' means 'even'. For both m and n are even, use $\sin^2 \alpha x = \frac{1}{2}(1 - \cos 2\alpha x)$ and $\cos^2 \alpha x = \frac{1}{2}(1 + \cos 2\alpha x)$ so that

$$I_{m,n}(\alpha) = 2^{-\frac{m+n}{2}} \sum_{r=0}^{m/2} \sum_{s=0}^{n/2} (-1)^r C_r^{m/2} C_s^{n/2} I_{0,r+s}(2\alpha).$$

6. $I_{m,n} = \int \tan^m \alpha x \sec^n \alpha x dx$, where $m, n \in \mathbb{Z}$, $m, n \geq 0$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

(m, n)	Method	$I_{m,n} =$
(o, $\cdot$), $n > 0$	Sub.: $u = \sec \alpha x$	$\frac{1}{\alpha} \sum_{r=0}^{(m-1)/2} C_r^{(m-1)/2} \frac{(-1)^r \sec^{m+n-2r-1} \alpha x}{m+n-2r-1} + C$
($\cdot$, e), $n > 0$	Sub.: $u = \tan \alpha x$	$\frac{1}{\alpha} \sum_{r=0}^{(n-2)/2} C_r^{(n-2)/2} \frac{\tan^{m+2r+1} \alpha x}{m+2r+1} + C$

where 'o' means 'odd' while 'e' means 'even'. For all other cases, use integration by parts and $\tan^2 \alpha x = \sec^2 \alpha x - 1$ to obtain the following reduction formulae:

$$I_{m,n} = -\frac{m-1}{m+n-1} I_{m-2,n} + \frac{\tan^{m-1} \alpha x \sec^n \alpha x}{\alpha(m+n-1)}, \quad \text{for } m \geq 2,$$
$$I_{m,n} = \frac{n-2}{m+n-1} I_{m,n-2} + \frac{\tan^{m+1} \alpha x \sec^{n-2} \alpha x}{\alpha(m+n-1)}, \quad \text{for } n \geq 2,$$
$$I_{0,1} = \frac{1}{\alpha} \ln|\sec \alpha x + \tan \alpha x| + C; \quad I_{1,0} = \frac{1}{\alpha} \ln|\sec \alpha x| + C; \quad I_{0,0} = C.$$

Problems that may be demonstrated in class :

Q1. Let $m \in \mathbb{N}$. Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^m}{n^{m+1}}$.

Q2. Show that $\ln \frac{n+1}{n} \leq \frac{1}{n}$ for any positive integer n. Hence use comparison test to prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Q3. Compute the following indefinite/definite integrals:

(a) $\int x^3 e^{2x} dx$; (b) $\int_1^e x^{-2}(\ln x)^2 dx$; (c) $\int_0^1 x^2 \sin \pi x dx$; (d) $\int x \sin 7x \cos 2x dx$;

(e) $\int \sec^4 x dx$; (f) $\int \sin^2 x \cos^2 x dx$; (g) $\int_0^{\pi/2} \sin^5 x dx$; (h) $\int_{-1}^{1} \sinh x \sec^3 x dx$;

(i) $\int \arcsin x dx$; (j) $\int \sinh x \cos x dx$; (k) $\int e^x \sin 3x dx$; (l) $\int \tan^2 x \sec^3 x dx$.

Q4. $\forall n \in \mathbb{Z}$ with $n \geq 0$, let $I_n = \int (\arcsin x)^n dx$ and $J_n = \int_0^1 (\arcsin x)^n dx$.

(a) Prove that $I_{n+2} = x(\arcsin x)^{n+2} + (n+2)\sqrt{1-x^2}(\arcsin x)^{n+1} - (n+2)(n+1)I_n$;

(b) Prove that

$$J_n = \begin{cases} \sum_{r=0}^{n/2} (-1)^{\frac{n}{2}-r} \frac{n!}{(2r)!} \left(\frac{\pi}{2}\right)^{2r}, & \text{if } n \text{ is even;} \\ (-1)^{\frac{n+1}{2}} n! + \sum_{r=0}^{(n-1)/2} (-1)^{\frac{n-1}{2}-r} \frac{n!}{(2r+1)!} \left(\frac{\pi}{2}\right)^{2r+1}, & \text{if } n \text{ is odd.} \end{cases}$$