Solutions to Homework IV

1. Firstly, we show that $u_t + u_x = 0$ in the region $A = \{t > b - x\}$. For (t_0, x_0) in A, let

$$z(s) = (u_t + u_x)(t_0 + s, x_0 - s)$$

for $s \in [x_0 - b, x_0 - a]$. Then

$$z'(s) = 0.$$

Since $z(x_0 - b) = 0$, $z(0) = (u_t + u_x)(t_0, x_0) = 0$. Next we show that $u = 0$ in the region $B = \{t > x + b - 2a\}$. For (t_0, x_0) in B, let

$$z(s) = u(t_0 + s, x_0 + s)$$

for $s \in [-x_0 + a, -x_0 + b]$. Then

$$z'(s) = 0$$

since $(t_0 + s, x_0 + s) \in A$ for $s \in [-x_0 + a, -x_0 + b]$. Since $z(-x_0 + a) = 0$, $z(0) = u(t_0, x_0) = 0$. When $t > 2(b - a)$, it is clear that $(t, x) \in B$, so $u(t, x) = 0$. By continuity, $u \equiv 0$ for $t \geq 2(b - a)$.

2. (a) Denote $\partial_i u$ by $u_{,i}$.

$$\partial_t^2 \mathbf{E} = \partial_t(\operatorname{curl} \mathbf{B}) = \operatorname{curl}(\partial_t \mathbf{B}) = -\operatorname{curl}(\operatorname{curl} \mathbf{E}).$$

Next we verify the following indentity:

$$\operatorname{curl}(\operatorname{curl} \mathbf{u}) = \nabla \operatorname{div} \mathbf{u} - \Delta \mathbf{u}.$$

We just verify the first entry here.

$$\begin{aligned} \operatorname{curl}(\operatorname{curl} \mathbf{u}) &= (\operatorname{curl} \mathbf{u})^3_{,2} - (\operatorname{curl} \mathbf{u})^2_{,3} \\ &= \mathbf{u}^2_{,12} - \mathbf{u}^1_{,22} - \mathbf{u}^1_{,33} + \mathbf{u}^3_{,13} \\ &= (\operatorname{div} \mathbf{u})_{,1} - \Delta \mathbf{u}^1. \end{aligned}$$

So

$$\partial_t^2 \mathbf{E} = -\nabla \operatorname{div} \mathbf{E} + \Delta \mathbf{E} = \Delta \mathbf{E}.$$

Similarly,

$$\partial_t^2 \mathbf{B} = -\operatorname{curl}(\partial_t \mathbf{E}) = -\operatorname{curl}(\operatorname{curl} \mathbf{B}) = -\nabla \operatorname{div} \mathbf{B} + \Delta \mathbf{B} = \Delta \mathbf{B}.$$

(b) Taking divergence, we have

$$\partial_t^2 w - \mu\Delta w - (\lambda+\mu)\Delta w = 0.$$

So w satisfies the wave equation:

$$\partial_t^2 w - (\lambda+2\mu)\Delta w = 0,$$

whose speed of propagation is $\sqrt{\lambda+2\mu}$.

Taking curl, we have

$$\partial_t^2 \mathbf{v} - \mu\Delta \mathbf{v} = 0.$$

So $\mathbf{v}$ satisfies the wave equation:

$$\partial_t^2 \mathbf{v} - \mu\Delta \mathbf{v} = 0,$$

whose speed of propagation is $\sqrt{\mu}$.

3. (a)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(k(t)+p(t)) &= \int_{\mathbb{R}} (u_t u_{tt} + u_x u_{xt}) \\ &= \int_{\mathbb{R}} (u_t u_{xx} + u_x u_{xt}) \\ &= 0.\end{aligned}$$

So $k(t)+p(t)$ is constant in t.

(b) Suppose that g and h are supported in $B(0,R)$. Then we show that $k(t)=p(t)$ when $t>R$. By d'Alembert's formula,

$$u(t,x) = \frac{g(x+t)+g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y.$$

Then,

$$\begin{aligned}u_t(t,x) &= \frac{g'(x+t)-g'(x-t)}{2} + \frac{h(x+t)+h(x-t)}{2}, \\ u_x(t,x) &= \frac{g'(x+t)+g'(x-t)}{2} + \frac{h(x+t)-h(x-t)}{2}.\end{aligned}$$

When $t>R$, $x+t$ and $x-t$ can't both lie in $B(0,R)$ since otherwise

$$2t \le |x+t| + |x-t| \le 2R.$$

So at least one of $(g'(x+t), h(x+t))$ and $(g'(x-t), h(x-t))$ vanishes. Hence, when $t>R$,

$$u_t^2 = u_x^2,$$

yielding that

$$k(t) = p(t).$$

4. Rewrite the equation as

$$u_{tt} + u_t + \frac{1}{4}u - u_{xx} + \frac{3}{4}u = 0.$$

By multiplying it by $\mathrm{e}^{t/2}$, we have

$$(\mathrm{e}^{t/2}u)_{tt} - (\mathrm{e}^{t/2}u)_{xx} + \frac{3}{4}(\mathrm{e}^{t/2}u) = 0.$$

Let $v = \mathrm{e}^{t/2}u$, then v satisfies the Klein-Gordon equation

$$v_{tt} - v_{xx} + \frac{3}{4}v = 0.$$

Next we establish the energy estimate for the Klein-Gordon equation. By multiplying it by v_t and integrating it on $\mathbb{R}$, we have

$$\int \left(v_{tt}v_t - v_{xx}v_t + \frac{3}{4}vv_t \right) = 0.$$

By integration by parts,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \left(v_t^2 + v_x^2 + \frac{3}{4}v^2 \right) = 0.$$

So

$$\widetilde{E}(t) = \int \left(v_t^2 + v_x^2 + \frac{3}{4}v^2 \right)$$

is constant and thus bounded. Next we return to u.

$$\begin{aligned}\widetilde{E}(t) &= \mathrm{e}^t \int \left[\left(\frac{u}{2} + u_t \right)^2 + u_x^2 + \frac{3}{4}u^2 \right] \\ &\geq \frac{3}{4}\mathrm{e}^t \int \left[\left(\frac{u}{2} + u_t \right)^2 + u_x^2 + u^2 \right].\end{aligned}$$

So

$$\begin{aligned}E(t) &= \int (u_t^2 + u_x^2 + u^2) \\ &\leq 2\int \left[\left(\frac{u}{2} + u_t \right)^2 + \left(\frac{u}{2} \right)^2 + u_x^2 + u^2 \right] \\ &\leq \frac{5}{2}\int \left[\left(\frac{u}{2} + u_t \right)^2 + u_x^2 + u^2 \right] \\ &\leq \frac{10}{3}\mathrm{e}^{-t}\widetilde{E}(t).\end{aligned}$$

Therefore,

$$\lim_{t\to\infty} E(t) = 0.$$