Solutions of Homework III

1. (i)

$$\begin{aligned}\Delta w &= 2u\Delta u + 2v\Delta v + 2|\nabla u|^2 + 2|\nabla v|^2 \\ &= -2(1-w)w + 2|\nabla u|^2 + 2|\nabla v|^2.\end{aligned}$$

So the equation w satisfies is

$$\Delta w = -2(1-w)w + 2|\nabla u|^2 + 2|\nabla v|^2. \tag{1}$$

(ii) Assume that w attains its maximum M at x_0. If $x_0 \in B(0,1)$, then $\Delta w(x_0) \leq 0$. By (1), $-(1-M)M \leq 0$. So $M \leq 1$. If $x_0 \in \partial B(0,1)$, then $M = 0 \leq 1$. Therefore $M \leq 1$.

2. (i) For the equations

$$\begin{cases} xf_x + yf_y = xy\log(xy); & (2) \\ x^2 f_{xx} + y^2 f_{yy} = xy, & (3) \end{cases}$$

consider $x(2)_x + y(2)_y$, we have

$$x^2 f_{xx} + y^2 f_{yy} + xf_x + yf_y + 2xyf_{xy} = 2xy(\log(xy) + 1).$$

Substituting (2) and (3) into it, we obtain that

$$f_{xy} = \frac{\log(xy) + 1}{2}.$$

(ii)

$$\begin{aligned} & f(s+1,s+1) - f(s+1,s) - f(s,s+1) + f(s,s) \\ = & \int_0^1 (f_y(s+1,s+q) - f_y(s,s+q))\,\mathrm{d}q \\ = & \int_0^1\int_0^1 f_{xy}(s+p,s+q)\,\mathrm{d}p\,\mathrm{d}q \\ = & \int_{[0,1]^2} \frac{\log[(s+p)(s+q)]+1}{2}\,\mathrm{d}p\,\mathrm{d}q. \end{aligned}$$

So

$$m(f) = \min_{s\geq 1}\int_{[0,1]^2} \frac{\log[(s+p)(s+q)]+1}{2}\,\mathrm{d}p\,\mathrm{d}q$$

$$
\begin{aligned}
&= \int_{[0,1]^2} \frac{\log[(1+p)(1+q)]+1}{2}\,\mathrm{d}p\,\mathrm{d}q \\
&= \frac{4\log 2 - 1}{2},
\end{aligned}
$$

and it is independent of f.

3. (i) By the strong maximum principle, since $0 \le u \le 1$ on the parabolic boundary and u is not a constant, $0 < u(t,x) < 1$ for all $(t,x) \in \mathbb{R}^+ \times (0,1)$.

   (ii) Since $u(t,x)$ and $u(t,1-x)$ are solutions to the heat equation and they agree on the parabolic boundary, by the uniqueness of initial boundary problems of heat equations, $u(t,x) = u(t,1-x)$ for all $t \ge 0$ and $0 \le x \le 1$.

   (iii) We have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u^2 &= \int_0^1 2uu_t \\
&= \int_0^1 2uu_{xx} \\
&= -\int_0^1 2u_x^2.
\end{aligned}
$$

   Moreover,

$$
\int_0^1 u_x^2 \neq 0.
$$

   In fact, if

$$
\int_0^1 u_x(t,x)^2\,\mathrm{d}x = 0,
$$

   then $u_x(t,x) = 0$ for all $0 \le x \le 1$. Since $u(t,0) = 0$, $u(t,x) = 0$ for all $0 \le x \le 1$, which contradicts the conclusion of (i). Therefore

$$
-\int_0^1 2u_x^2 < 0
$$

   and $\int_0^1 u^2$ is strictly decreasing.

4. (i) Since a solution to $u_t = u$ is e^t, we may consider $v = \mathrm{e}^{-t}u$. By substituting $u = \mathrm{e}^t v$ into the equation, we have

$$
\mathrm{e}^t(v_t + v) = \mathrm{e}^t v_{xx} + \mathrm{e}^t v.
$$

   So

$$
v_t - v_{xx} = 0.
$$

   Moreover, $v(0,x) = u(0,x) = \phi(x)$. Therefore,

$$
v(t,x) = \int S(t,x-y)\phi(y)\,\mathrm{d}y
$$

and

$$u(t,x) = \mathrm{e}^t \int S(t, x-y)\phi(y)\,\mathrm{d}y.$$

And it is easy to verify that the above u is a solution.

(ii) Since a solution to $u_t = t^2 u$ is $\mathrm{e}^{t^3/3}$, we may consider $v = \mathrm{e}^{-t^3/3}u$. By substituting $u = \mathrm{e}^{t^3/3}v$ into the equation, we have

$$\mathrm{e}^{t^3/3}v_t + t^2\mathrm{e}^{t^3}v = \mathrm{e}^{t^3/3}v_{xx} + t^2\mathrm{e}^{t^3/3}v.$$

So

$$v_t - v_{xx} = 0.$$

Moreover, $v(0,x) = u(0,x) = \phi(x)$. Therefore,

$$v(t,x) = \int S(t, x-y)\phi(y)\,\mathrm{d}y$$

and

$$u(t,x) = \mathrm{e}^{t^3/3} \int S(t, x-y)\phi(y)\,\mathrm{d}y.$$

And it is easy to verify that the above u is a solution.

(iii) Consider $v(t,x) = u(t, x-t)$. Then

$$v_t(t,x) = u_t(t,x-t) - u_x(t,x-t) = u_{xx}(t,x-t) = v_{xx}(t,x).$$

Moreover, $v(0,x) = u(0,x) = \phi(x)$. Therefore,

$$v(t,x) = \int S(t, x-y)\phi(y)\,\mathrm{d}y$$

and

$$u(t,x) = v(t, t+x) = \int S(t, t+x-y)\phi(y)\,\mathrm{d}y.$$

And it is easy to verify that the above u is a solution.

5. (i) Since v is a solution of the heat equation, w is also a solution of the heat equation.

$$\begin{aligned}
v_x(t,x) &= \frac{1}{\sqrt{4\pi t}} \int \partial_x \left(\mathrm{e}^{-\frac{(x-y)^2}{4t}}\right) f(y)\,\mathrm{d}y \\
&= -\frac{1}{\sqrt{4\pi t}} \int \partial_y \left(\mathrm{e}^{-\frac{(x-y)^2}{4t}}\right) f(y)\,\mathrm{d}y \\
&= -\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{0} \partial_y \left(\mathrm{e}^{-\frac{(x-y)^2}{4t}}\right) f(y)\,\mathrm{d}y - \frac{1}{\sqrt{4\pi t}} \int_{0}^{\infty} \partial_y \left(\mathrm{e}^{-\frac{(x-y)^2}{4t}}\right) f(y)\,\mathrm{d}y \\
&= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{0} \mathrm{e}^{-\frac{(x-y)^2}{4t}} f'(y)\,\mathrm{d}y - \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\frac{x^2}{4t}} f(0) \\
&\quad + \frac{1}{\sqrt{4\pi t}} \int_{0}^{\infty} \mathrm{e}^{-\frac{(x-y)^2}{4t}} f'(y)\,\mathrm{d}y + \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\frac{x^2}{4t}} f(0)
\end{aligned}$$

$$= \frac{1}{\sqrt{4\pi t}} \int \mathrm{e}^{-\frac{(x-y)^2}{4t}} f'(y)\,\mathrm{d}y.$$

So

$$v_x - 2v = \frac{1}{\sqrt{4\pi t}} \int \mathrm{e}^{-\frac{(x-y)^2}{4t}} (f'(y) - 2f(y))\,\mathrm{d}y.$$

Since $f' - 2f$ is continuous on $\mathbb{R} \setminus \{0\}$ and $(f'-2f)(0+) = 1$ and $(f'-2f)(0-) = -1$,

$$w(0,x) = \begin{cases} 1 - 2x & x > 0; \\ 0 & x = 0; \\ -1 - 2x & x < 0. \end{cases}$$

(ii) It is clear that

$$f'(x) - 2f(x) + f'(-x) - 2f(-x) = 0$$

for all $x \neq 0$. So $f' - 2f$ is an odd function for $x \neq 0$.

(iii) Let $g = f' - 2f$. Then

$$w(t,x) = \frac{1}{\sqrt{4\pi t}} \int \mathrm{e}^{-\frac{y^2}{4t}} g(x-y)\,\mathrm{d}y.$$

$$\begin{aligned} w(t,-x) &= \frac{1}{\sqrt{4\pi t}} \int \mathrm{e}^{-\frac{y^2}{4t}} g(-x-y)\,\mathrm{d}y \\ &= \frac{1}{\sqrt{4\pi t}} \int \mathrm{e}^{-\frac{y^2}{4t}} g(-x+y)\,\mathrm{d}y \\ &= -\frac{1}{\sqrt{4\pi t}} \int \mathrm{e}^{-\frac{y^2}{4t}} g(x-y)\,\mathrm{d}y \\ &= -w(t,x). \end{aligned}$$

Hence $w(t,x)$ is an odd function of x.

(iv) It suffices to prove that

$$v_x(t,0) - 2v(t,0) = 0$$

for $t > 0$. Since w is an odd function of x, it holds.

6. Solve the ODE

$$\begin{cases} f'(x) - hf(x) = -\phi'(-x) + h\phi(-x) \\ f(0) = \phi(0). \end{cases}$$

on $(-\infty, 0]$. Denote the solution

$$\phi(0)\mathrm{e}^{hx} + \int_0^x \mathrm{e}^{h(x-y)}(-\phi'(-y) + h\phi(-y))\,\mathrm{d}y$$

by $g(x)$. Then let

$$\widetilde{f}(x) = \begin{cases} \phi(x) & x \geq 0; \\ g(x) & x < 0. \end{cases}$$

and

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int \mathrm{e}^{-\frac{(x-y)^2}{4t}} \widetilde{f}(y)\,\mathrm{d}y.$$

Similar to Exercise 5, we could verify that u is a solution to the problem.